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Multiple Singular Emission in Gauge Theories

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Abstract

I derive a class of functions unifying all singular limits for the emission of a given number of soft or collinear gluons in tree-level gauge-theory amplitudes. Each function is a generalization of the single-emission antenna function of ref. [1]. The helicity-summed squares of these functions are thus also generalizations to multiple singular emission of the Catani–Seymour dipole factorization function.

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1. Introduction

The computation of higher-order corrections in perturbative QCD is important to the program of high-energy collider experiments, particularly at the Tevatron and LHC. Such computations involve a variety of technical complications, including the need to handle what would be a large number of diagrams in a conventional Feynman diagram approach. In the past decade, a number of new approaches have been developed to cope with this complexity, including the color decomposition [2], ideas based on string theory [3], and the unitarity-based method [4]. The latter technique has been applied to numerous calculations, most recently the two-loop calculation of all helicity amplitudes for gluon–gluon scattering [5,6].

The subject of two-loop calculations has seen tremendous progress in the last three years. Smirnov [7] gave a closed-form expression for the planar double box, and Tausk [8] one for the non-planar integral. Smirnov and Veretin [9] and Anastasiou et al. [10] provided algorithms for reducing tensor integrals. (More general reduction and evaluation techniques for integrals have followed as well [11].) These computations, along with the other integrals required for two-loop amplitudes [10,12] has in turn led to a long series of computations of four-point amplitudes [13,6]. These amplitudes and matrix elements are one of the building blocks of next-to-next-leading order (NNLO) computations in perturbative QCD, in particular of the cornerstone processes $e^+e^- \to$ 3 jets and $p\overline{p} \to 2$ jets.

In order to construct numerical programs from these and other amplitudes, we must confront another class of technical complications, that of handling infrared divergences. In the framework of dimensional regularization, gauge-theory loop amplitudes have poles of infrared origin in the regulator ϵ, up to two powers per loop. These poles are canceled in physical differential cross sections by divergences arising from integrations over infrared-singular regions of the phase space for real emission of additional partons from corresponding lower-loop amplitudes.

At next-to-leading order, we need to consider two types of singularities, soft and collinear. The former arises when a gluon four-momentum vanishes, $k_s \to 0$; the latter when the momenta of two massless particles become proportional, $k_a \to z(k_a + k_b)$, $k_b \to (1 - z)(k_a + k_b)$. It is helpful to combine these two into a single function describing both limits, as proposed by Catani and Seymour [14] for the square of the matrix element. I wrote down an *antenna* function or amplitude [1] providing a similar unification at the amplitude level, within the framework of a color decomposition.

At next-to-next-to-leading order, in addition to single-emission singularities in one-loop amplitudes, we must now handle several types of double-emission singularities in tree level amplitudes, with correspondingly complicated internal boundaries in phase space: double-soft [15], soft-collinear [16,17], and triple-collinear [16,17,18]. The aim of the present paper is to extend the notions of ref. [1] to double- and multiple-singular emission, and to provide a single function describing the factorization of a color-ordered amplitudes in all the different singular limits of a color-sequential set of momenta. The integrals of such functions over phase space will provide the appropriate generalization of the integrated Catani–Seymour functions to NNLO computations. It is worth noting that Catani has predicted [19] the IR poles to be expected in the pure two-loop virtual corrections, which must be canceled by the sum of the double-emission amplitudes and the single-emission one-loop amplitudes.

The properties of non-Abelian gauge-theory amplitudes in singular limits are easiest to understand in the context of a color decomposition [2]. In the present paper, I will concentrate on tree-level all-gluon amplitudes, though the formalism readily extends to amplitudes with quarks and (colored) scalars as well. For tree-level all-gluon amplitudes in an $SU(N)$ gauge theory the color decomposition has the form,

$$\mathcal{A}_n^{\rm tree}(\{k_i,\lambda_i,a_i\}) = \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\, A_n^{\rm tree}(\sigma(1^{\lambda_1},\ldots,n^{\lambda_n}))\,, \tag{1.1}$$

where S_n/Z_n is the group of non-cyclic permutations on n symbols, and j^{λ_j} denotes the j-th momentum and helicity. As is by now standard, I use the normalization ${\rm Tr}(T^aT^b)=\delta^{ab}$. One can write analogous formulæ for amplitudes with quark-antiquark pairs or uncolored external lines. The color-ordered or partial amplitude A_n is gauge invariant, and has simple factorization properties in both the soft and collinear limits,

$$\begin{aligned}
A_n^{\rm tree}(\ldots,a^{\lambda_a},b^{\lambda_b},\ldots) &\xrightarrow{a\|b} \sum_{\lambda=\pm} {\rm C}_{-\lambda}^{\rm tree}(a^{\lambda_a},b^{\lambda_b};z)\, A_{n-1}^{\rm tree}(\ldots,(a+b)^{\lambda},\ldots)\,,\\
A_n^{\rm tree}(\ldots,a,s^{\lambda_s},b,\ldots) &\xrightarrow{k_s\to 0} {\rm Soft}^{\rm tree}(a,s^{\lambda_s},b)\, A_{n-1}^{\rm tree}(\ldots,a,b,\ldots)\,.
\end{aligned} \tag{1.2}$$

The collinear splitting amplitude ${\rm C}^{\rm tree}$, squared and summed over helicities, gives the usual unpolarized Altarelli–Parisi splitting function [20]. It depends on the collinear momentum fraction z (here made explicit) in addition to invariants built out of the collinear momenta. While the complete amplitude also factorizes in the collinear limit, the same is not true of the soft limit; the eikonal factors ${\rm Soft}^{\rm tree}$ get tangled up with the color structure. It is for this reason that the color decomposition is useful.

I will review the unification of these functions into the single-emission antenna amplitude in section `\SingleEmissionSection`. The derivations and formalism of the present paper are based on use of recurrence relations for gauge-theory amplitudes, which I review in the next section. In section \MultipleCollinearSection, I consider a subtlety that arises in the derivation of factorization functions for multiply-collinear emission; the collinear splitting amplitudes for $1 \to 2$ and $1 \to 3$ configurations are given in the appendix. The use of a unified factorization requires *reconstruction* functions describing the factorized hard legs; I review the $3 \to 2$ reconstruction functions in section \SingleReconstructionSection, and describe the generalization to $n \to 2$ in section \GeneralReconstructionSection. I construct the antenna amplitude for double singular emission in section \DoubleEmissionSection, and that for multiple singular emission in section \MultipleEmissionSection. I give explicit expressions for specific helicities of the single- and double-emission functions in section \DoubleEmissionHelicitySection. For use in integrations over singular phase space, it is convenient to have the squares of the antenna amplitudes in dimensional regularization. I provide such expressions for the squares of the single- and double-emission antenna amplitudes in section \DimensionalRegularizationSection.

2. Recurrence Relations

I will base the derivations in this paper on the recurrence relations formalism of Berends and Giele [21], starting with the form given by Dixon [22] with a slightly different notation. (For a helicity-based form of recurrence relations, see ref. [23].)

The recurrence relations define an n-point color-ordered gluon current, $J^\mu(1,\ldots,n{-}1;y)$, with the leg indexed by μ off shell, and legs $1,\ldots,n{-}1$ on shell. In Dixon's notation, $J^\mu(1,\ldots,n{-}1;0)$ here is $-J^\mu(1,\ldots,n{-}1)$. All momenta are taken to be outgoing, and I have introduced an additional argument y representing a momentum excess, $k_y = -(k_1 + \cdots + k_n + k_x)$, for reasons which will become clear in later sections. For now, the reader may imagine that this additional argument is always zero. I will use labels interchangeably with momenta carrying those labels as arguments. For later use, it will also be convenient to define a contracted form making the last momentum an explicit argument, $J(1,\ldots,n;P;y) \equiv \varepsilon_P \cdot J(1,\ldots,n;y)$.

Define $K_{i,j} = k_i + \cdots + k_j$; the recurrence relations then have the form,

$$
\begin{aligned}
&J_\mu(1,\ldots,n;k_y) = \\
&-\frac{i d_{\mu\mu'}(K_{1,n})}{K_{1,n}^2}\left[\sum_{j=1}^{n-1} V_3^{\mu'\nu'\rho'}(K_{1,j}+\zeta_1 k_y, K_{j+1,n}+\zeta_2 k_y, -K_{1,n}+\zeta_3 k_y) d_{\nu\nu'}(K_{1,j}) d_{\rho\rho'}(K_{j+1,n})\right. \\
&\qquad\qquad \times\, J^\nu(1,\ldots,j;\alpha_1 k_y) J^\rho(j+1,\ldots,n;\alpha_2 k_y) \\
&\quad - \sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n-1} V_4^{\mu'\nu'\rho'\lambda'} d_{\nu\nu'}(K_{1,j_1}) d_{\rho\rho'}(K_{j_1+1,j_2}) d_{\rho\rho'}(K_{j_2+1,n}) \\
&\qquad\qquad \left.\times\, J^\nu(1,\ldots,j_1;\beta_1 k_y) J^\rho(j_1+1,\ldots,j_2;\beta_2 k_y) J^\lambda(j_2+1,\ldots,n;\beta_3 k_y)\right]
\end{aligned}
\tag{2.1}
$$

where the three-point vertex is

$$
V_3^{\mu\nu\rho}(P_1,P_2,P_3) = \frac{i}{\sqrt{2}}\left[g^{\nu\rho}(P_1-P_2)^\mu + 2g^{\rho\mu}P_2^\nu - 2g^{\mu\nu}P_1^\rho\right] , \tag{2.2}
$$

the four-point vertex is

$$
V_4^{\mu\nu\rho\lambda} = \frac{i}{2}\left[2g^{\mu\rho}g^{\nu\lambda} - g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\nu\rho}\right] , \tag{2.3}
$$

and $d_{\mu\nu}$ is the gluon helicity projector; in the background-field form used here, $d_{\mu\nu} = -g_{\mu\nu}$. (Note that the current as defined here has the opposite sign to that of ref. [22].) Define as well $J(1;P;k_y) = \varepsilon_P \cdot \varepsilon_1$. A sum over intermediate polarizations will be understood implicitly in all products of indexless currents or currents and amplitudes in this paper,

$$
J(\ldots;P;y)X(\ldots,-P,\ldots) = \sum_{\text{polarizations } \sigma} J(\ldots;P^\sigma;y)X(\ldots,-P^{-\sigma},\ldots) \tag{2.4}
$$

The n-point amplitude $A_n^{\text{tree}}(1,\ldots,n)$ is given by removing the propagator on the last, off-shell, leg of the contracted current, and then taking the on-shell limit,

$$
A_n^{\text{tree}}(1,\ldots,n) = \lim_{k_n^2\to 0} -i k_n^2 J(1,\ldots,n-1;n;0) \tag{2.5}
$$

3. Splitting Amplitudes for Multiple Collinear Emission

With the currents in hand, we are ready to begin the derivation of collinear splitting amplitudes. Let us begin by rederiving the collinear behavior of amplitudes as two momenta become collinear; without loss of generality, we may take these to be k_1 and k_2. We are interested in the limit when s_{12} becomes small compared to all other invariants in an n-point amplitude (or equivalently, shrinks compared to the dot products of an arbitrary reference momentum q with k_1 and k_2 [14]).

The limit will be dominated by contributions which have an explicit pole in this invariant. The structure of the recurrence relations tells us that the invariant always appears as a single pole, inside $J(1,2;-(k_1+k_2);0)$. We can also see that if we replace this current by a polarization vector carrying the fused momentum $k_P=-(k_1+k_2)$, we will obtain an $(n-1)$-point amplitude from eqn. (2.5). The original amplitude thus factorizes in this limit,

$$A_n^{\rm tree}(1,2,\ldots,n) \xrightarrow{s_{12}\to 0} J(1,2;k_P)A_{n-1}^{\rm tree}(1+2,3,\ldots,n)\,. \tag{3.1}$$

In this equation, the notation '$1+2$' is a shorthand for k_1+k_2. Gauge invariance ensures that only physical polarizations appear for the fused leg, but we do need to sum over both,

$$A_n^{\rm tree}(1,2,\ldots,n) \xrightarrow{s_{12}\to 0} \sum_{\text{ph. pol. }\sigma_P} \mathrm{C}^{\rm tree}_{\sigma_P}(1^{\sigma_1},2^{\sigma_2})A_{n-1}^{\rm tree}((1+2)^{-\sigma_P},3,\ldots,n)+\text{finite}\,. \tag{3.2}$$

The tree splitting amplitude is given by the singular limit of the current,

$$\mathrm{C}^{\rm tree}_{\sigma_P}(1^{\sigma_1},2^{\sigma_2}) = J(1^{\sigma_1},2^{\sigma_2};k_P^{\sigma_P};0)\big|_{\text{leading in } s_{12}} \tag{3.3}$$

The helicity algebra will soften the full pole in s_{12} to a square-root singularity most conveniently expressed in terms of spinor products, so that the splitting amplitude will scale as $1/\sqrt{\delta}$ when the invariant shrinks by a factor of δ. The leading, singular, part is gauge-invariant, in spite of the seemingly off-shell nature of this object.

To derive a similar factorization and function when three color-adjacent external momenta k_1, k_2, and k_3 become collinear simultaneously, we must extract all singularities in s_{12}, s_{23}, and $t_{123}=s_{12}+s_{23}+s_{13}$, with these invariants all of comparable order and much smaller than other invariants of external momenta. (The strongly-ordered limit where some of the two-particle invariants are much smaller again than t_{123} is contained as a degenerate case in the limit of interest; what we are *not* interested in is the limit where t_{123} becomes much smaller than any of the two-particle invariants s_{12}, s_{23}, or s_{13}.) Color-ordered amplitudes will not have singularities as the invariants of non-adjacent legs, such as s_{13}, become collinear. Furthermore, as explained by Campbell and Glover [16], we are interested only in the leading behavior, corresponding to singularities in *two* final-state integration variables, or equivalently that scale as $1/\delta$ (the square of the scaling above) when all the above invariants shrink by a factor of δ.

We begin by extracting all singularities in t_{123}. This invariant appears only in the four-point current $J(1,2,3;P)$. Next, include the invariants s_{12} and s_{23}, which appear only in $J(1,2;P)$ and $J(2,3;P)$ respectively. We might thus be tempted to write down the following starting formula,

$$\begin{aligned} A_n^{\rm tree}(1,2,\ldots,n) \xrightarrow{1\|2\|3} & J(1,2,3;k_P)A_{n-2}^{\rm tree}(1+2+3,\ldots,n) \\ & + J(1,2;k_P)A_{n-1}^{\rm tree}(1+2,3,\ldots,n) \\ & + J(2,3;k_P)A_{n-1}^{\rm tree}(1,2+3,\ldots,n)\,, \end{aligned} \tag{3.4}$$

but this double-counts certain contributions, because $J(1,2,3;P)$ *also* contains poles in s_{12} and s_{23}, and $A_{n-1}^{\rm tree}$ contains a pole in t_{123}. Subtract off the double-counted contribution, to obtain the corrected formula,

$$
\begin{aligned}
A_n^{\rm tree}(1,2,\ldots,n) \xrightarrow{1\|2\|3} & J(1,2,3;k_P)A_{n-2}^{\rm tree}(1+2+3,\ldots,n) \\
& + J(1,2;k_P)A_{n-1}^{\rm tree}(1+2,3,\ldots,n) \\
& + J(2,3;k_P)A_{n-1}^{\rm tree}(1,2+3,\ldots,n) \\
& - J(1,2,3;k_P)\big|_{s_{12}\ \rm pole}\, A_{n-2}^{\rm tree}(1+2+3,\ldots,n) \\
& - J(1,2,3;k_P)\big|_{s_{23}\ \rm pole}\, A_{n-2}^{\rm tree}(1+2+3,\ldots,n)\,,
\end{aligned}
\tag{3.5}
$$

We are not done, however, because the $(n-1)$-point amplitudes appearing here contain additional singularities in the limit. These may again be expressed in terms of the three-point current,

$$
\begin{aligned}
A_n^{\rm tree}(1,2,\ldots,n) \xrightarrow{1\|2\|3} & J(1,2,3;k_P)A_{n-2}^{\rm tree}(1+2+3,\ldots,n) \\
& + J(1,2;k_R)J(1+2,3;k_P)A_{n-2}^{\rm tree}(1+2+3,\ldots,n) \\
& + J(2,3;k_R)J(1,2+3;k_P)A_{n-2}^{\rm tree}(1+2+3,\ldots,n) \\
& - J(1,2,3;k_P)\big|_{s_{12}\ \rm pole}\, A_{n-2}^{\rm tree}(1+2+3,\ldots,n) \\
& - J(1,2,3;k_P)\big|_{s_{23}\ \rm pole}\, A_{n-2}^{\rm tree}(1+2+3,\ldots,n)\,,
\end{aligned}
\tag{3.6}
$$

I emphasize that despite the appearance of sequential factorization in the second and third terms, no assumption is made about strong ordering of the limits; for example, s_{12} is *not* taken to be much smaller than s_{23} or t_{123} in the second term.

The subtlety which this section addresses arises from the fact that the last two terms in eqn. (3.5) are not gauge-invariant (because $k_P^2 \neq 0$). To see this explicitly, it is convenient to decompose the propagator giving rise to (say) the s_{12} pole in a helicity basis [24],

$$
\begin{aligned}
J(1,2,3;P)\big|_{s_{12}\ \rm pole} = & J^\mu(1,2;-K_{1,2}) \\
& \times \Big[\varepsilon_\mu^{(+)}(-K_{1,2};q)\varepsilon_{\mu'}^{(-)}(K_{1,2};q) + \varepsilon_\mu^{(-)}(-K_{1,2};q)\varepsilon_{\mu'}^{(+)}(K_{1,2};q) \\
& - \frac{(K_{1,2})_\mu q_{\mu'}}{q\cdot K_{1,2}} - \frac{q_\mu (K_{1,2})_{\mu'}}{q\cdot K_{1,2}}\Big] V_3^{\mu'\nu\rho}(-K_{1,2},3,P) J_\nu(3;-k_3)\varepsilon_{P\rho}
\end{aligned}
\tag{3.7}
$$

where q is a null vector not collinear to any of the external momenta, and $\varepsilon_\mu^{(\sigma)}(K;q)$ is a polarization vector for a gluon of helicity σ carrying momentum K, with reference momentum q.

The first two terms in brackets sum to the second term in eqn. (3.6), and cancel it. A similar cancellation occurs for the third term in eqn. (3.6). The third term in eqn. (3.7) will vanish by gauge invariance, $K_{1,2}\cdot J(1,2;P)=0$. The last term is new, and survives; indeed, here the helicity algebra in $q \cdot J(1,2;P)$ will not soften the s_{12} pole, although $K_{1,2} \cdot V_3$ will give rise to a factor of $k_P^2 = t_{123}$ that will cancel off the t_{123} pole, so that overall this term will scale like $1/\delta$.

The equation above is given in Feynman gauge. If we identify q as the light-cone vector of light-cone gauge, then the last two terms correspond precisely to the difference between the spin-projectors of the gluon propagators in Feynman and light-cone gauges. That is, were we to work in light-cone gauge, the last two terms in eqn. (3.7) would be absent, and the splitting amplitude would be given (as we might naively have expected) by the singular limit of the four-point current $J(1,2,3;P)$ alone. (A similar point about the convenience of a physical gauge for derivations of collinear limits was made by Catani and Grazzini [17] in the context of a different formalism, and by Dixon [25].)

In light-cone gauge, the three-vertex has the form,

$$V_{3,\mathrm{LC}}^{\mu\nu\rho}(P_1,P_2,P_3) = \frac{i}{\sqrt{2}}\left[g^{\nu\rho}(P_1-P_2)^\mu + g^{\rho\mu}(P_2-P_3)^\nu + g^{\mu\nu}(P_3-P_1)^\rho\right], \tag{3.8}$$

the four-point vertex is unchanged, the gluon propagator's helicity projector is,

$$d_{\mu\nu}(k) = -g_{\mu\nu} + \frac{q^\mu k^\nu + k^\mu q^\nu}{q \cdot k} \tag{3.9}$$

(where q is the light-cone momentum), and the recurrence relations themselves have the same form as in eqn. (2.1). All currents appearing in the following sections are understood to be in light-cone gauge.

Campbell and Glover extracted [16] the helicity-summed squared triple-collinear splitting function from squared amplitudes. Catani and Grazzini computed [17] a somewhat more general object, retaining the dependence on the polarization of the parent parton, but still at the level of the squared matrix element rather than at the amplitude level. Neither set of authors provided a factorizing form at the amplitude level. Del Duca, Frizzo, and Maltoni [18] derived splitting amplitudes from six-point amplitudes. The splitting amplitudes which follow from the light-cone current above are given in the appendix. They retain all relative phase and helicity-correlation information, and agree with the expressions given in ref. [18], up to a difference due to normalization conventions, and a sign for the amplitudes with two positive and two negative helicities. (The overall phase of these splitting amplitudes depends on the phase convention for the amplitudes; I follow the convention given by the Mangano and Parke review article [26].)

Squaring the splitting amplitudes, and summing over all helicities, I find a squared splitting function in agreement with the $\epsilon \to 0$ limit of the expression given by Campbell and Glover [16] (up to a constant factor related to the different normalization of amplitudes).

4. The Single-Emission Antenna Amplitude

As described in the introduction, gauge-theory amplitudes have singularities not only when color-adjacent momenta become collinear, of course, but also when external gluon momenta become soft. Factorization in both limits is given by universal functions [26]. Catani and Seymour pointed out that for computational purposes, it is desirable to unify the two different limits. They wrote down a function and a formalism for doing this for the color-summed squared amplitude, for lone singular emission (that is, one collinear pair or one soft gluon). In a previous paper [1], I constructed a function which unifies these two limits at the level of the color-ordered amplitude.

The construction is based on the observation that one may treat singular emission in gauge theories as occuring inside the 'color antenna' bounded by two hard colored particles, which I shall label a and b. For the single-emission function, we need to consider the emission of a single gluon carrying momentum k_1 inside the antenna. There are three sorts of singularities we should consider:

(a) $s_{a1} \to 0$, with s_{1b} approaching a constant non-zero limit, that is the collinear limit $k_1 \parallel k_a$;
(b) $s_{1b} \to 0$, with s_{a1} approaching a constant non-zero limit, that is the collinear limit $k_1 \parallel k_b$;
and (c) s_{a1} and $s_{1b} \to 0$, that is the soft limit $k_1 \to 0$.

In all cases, s_{ab} defines a hard scale; in the two collinear regions, the non-collinear hard momentum acts as a reference momentum to define the collinear momentum fractions.

To extract the antenna amplitude, we begin by extracting all contributions in the n-point amplitude that have a pole in either s_{a1} or s_{1b},

$$J(a,1;-(k_a+k_1);0)A_{n-1}(\ldots,k_a\!+\!k_1,b,\ldots) + J(1,b;-(k_a+k_1);0)A_{n-1}(\ldots,a,k_b\!+\!k_1,\ldots) \tag{4.1}$$

We introduce a complete set of polarization states for the unfused leg in each term, rewrite the product of polarization vectors as a two-point current, and introduce new labels for the surviving hard momenta,

$$\begin{aligned} &J(a,1;-(k_a\!+\!k_1);0)J(b;-k_b;0)A_{n-1}(\ldots,-k_{\hat a}=k_a\!+\!k_1,-k_{\hat b}=k_b,\ldots)\\ &+J(a;-k_a;0)J(1,b;-(k_1\!+\!k_b);0)A_{n-1}(\ldots,-k_{\hat a}=k_a,-k_{\hat b}=k_b\!+\!k_1,\ldots). \end{aligned} \tag{4.2}$$

At this point, the hatted momenta have different definitions in the two terms. If we can find a suitable pair of *reconstruction* functions $k_{\hat a,\hat b} = f_{\hat a,\hat b}(k_a,k_1,k_b)$, however, we can combine the two terms,

$$\begin{aligned} &[J(a,1;-(k_a\!+\!k_1);0)J(b;-k_b;0)+J(a;-k_a;0)J(1,b;-(k_1\!+\!k_b);0)]\\ &\qquad\times A_{n-1}(\ldots,-k_{\hat a}=f_{\hat a}(k_a,k_1,k_b),-k_{\hat b}=f_{\hat b}(k_a,k_1,k_b),\ldots). \end{aligned} \tag{4.3}$$

We also would like to have a manifestly gauge-invariant object inside the brackets, or equivalently get rid of the implicit dependence on the reference momentum q, so that we have a well-defined function in between the different singular limits as well. To do so, we would like to put the off-shell arguments in the currents on-shell. We can do so, at the price of violating momentum conservation within each current,

$$\begin{aligned}&[J(a,1;\hat{a};k_b{+}k_{\hat{b}})J(b;\hat{b};k_{\hat{a}}{+}k_a{+}k_1) + J(a;\hat{a};k_{\hat{b}}{+}k_b{+}k_1)J(1,b;\hat{b};k_a{+}k_{\hat{a}})]\\ &\quad\times A_{n-1}(\ldots,-k_{\hat{a}} = f_{\hat{a}}(k_a,k_1,k_b), -k_{\hat{b}} = f_{\hat{b}}(k_a,k_1,k_b),\ldots)\,.\end{aligned} \tag{4.4}$$

(It is for this purpose that I added the additional argument to the current in section 2.) The factor in brackets is the antenna factorization amplitude,

$$\mathrm{Ant}(\hat{a},\hat{b} \leftarrow a,1,b) = J(a,1;\hat{a};k_b{+}k_{\hat{b}})J(b;\hat{b};k_a{+}k_1{+}k_{\hat{a}}) + J(a;\hat{a};k_b{+}k_1{+}k_{\hat{b}})J(1,b;\hat{b};k_a{+}k_{\hat{a}})\,, \tag{4.5}$$

with corresponding factorization,

$$A_n(\ldots,a,1,b,\ldots) \xrightarrow{k_1\ \mathrm{singular}} \sum_{\mathrm{ph.\ pol.}\lambda_{a,b}} \mathrm{Ant}(\hat{a}^{\lambda_a},\hat{b}^{\lambda_b} \leftarrow a,1,b) A_{n-1}(\ldots,-k_{\hat{a}}^{-\lambda_a},-k_{\hat{b}}^{-\lambda_b},\ldots). \tag{4.6}$$

While there is a momentum excess inside each of the pair of currents making up a term, the excesses cancel so that overall, each term and hence the factor in brackets as a whole conserves momentum,

$$-(k_{\hat{a}} + k_{\hat{b}}) = k_a + k_1 + k_b \equiv K\,. \tag{4.7}$$

The reader might nonetheless wonder about the effect of this momentum excess on the uniqueness of the result; this question will be addressed in the following section.

The antenna amplitude describes the behavior of the amplitude when either $s_{1a}/s_{ab} \to 0$, or $s_{1b}/s_{ab} \to 0$, or both; equivalently, when $\Delta(a,1,b)/s_{ab}^3 \to 0$, with $\Delta(p_1,\ldots,p_n)$ the Gram determinant,

$$\Delta(p_1,\ldots,p_n) = \det(2p_i\cdot p_j) \tag{4.8}$$

(Note that the normalization here is non-standard.) I will also make use of the generalized Gram determinant,

$$G\begin{pmatrix} p_1,\ldots,p_n \\ q_1,\ldots,q_n \end{pmatrix} = \det(2p_i\cdot q_j), \tag{4.9}$$

which vanishes whenever two p_i or two q_i become collinear, or when any momentum becomes soft.

We still need to specify the reconstruction functions $f_{\hat{a},\hat{b}}$, however; that is the subject of the next section.

5. Single-Emission Reconstruction Function

What makes a suitable reconstruction function? Sensible functions will ensure that the reconstructed momenta are on shell, $k_{\hat{a}}^2 = 0 = k_{\hat{b}}^2$, as well as enforcing momentum conservation,

$k_{\hat a} + k_{\hat b} + k_a + k_1 + k_b = 0$. They must reduce to the appropriate forms at the singular points,

$$\begin{aligned} k_{\hat a} &= -(k_a+k_1), k_{\hat b} = -k_b, \qquad \text{when } s_{a1} = 0, s_{1b} \neq 0\,; \\ k_{\hat a} &= -k_a, k_{\hat b} = -(k_1+k_b), \qquad \text{when } s_{a1} \neq 0, s_{1b} = 0\,; \\ k_{\hat a} &= -k_a, k_{\hat b} = -k_b, \qquad \text{when } s_{a1} = 0 = s_{1b}\,. \end{aligned} \tag{5.1}$$

Finally, they must ensure that the excess momentum within each of the currents in eqn. (4.5) does not lead to additional terms which are singular in any of the limits.

To understand this last requirement better, examine the current $J(a, 1; \hat a; k_b + k_{\hat b})$. Because momentum is not conserved in this expression, different forms of the three-vertex – for example, the form given in eqn. (3.8) and (say),

$$\frac{i}{\sqrt{2}} \left[g^{\nu\rho}(P_1 - P_2)^\mu + g^{\rho\mu}(2P_2 + P_1)^\nu + g^{\mu\nu}(-P_2 - 2P_1)^\rho \right] , \tag{5.2}$$

will lead to different results. However, the results will be equivalent in the singular limits so long as they differ only by terms proportional to s_{a1}. When divided by the s_{a1} in the current, these will give rise only to finite terms. Such finite terms are anyway not universal, and are implicitly omitted from the antenna amplitude.

The difference will be a sum of terms, each proportional to $(k_b+k_{\hat b}) \cdot \varepsilon_{a,1,\hat a}$ or $(k_b+k_{\hat b}) \cdot (k_a-k_1)$, each of which should be of order s_{a1} (or higher) as $s_{1a} \to 0$.

The first two requirements, for null momenta and momentum conservation, can be satisfied by the following general form [1],

$$\begin{aligned} k_{\hat a} &= -\frac{1}{2(K^2 - s_{1b})} \left[(1+\rho)K^2 - 2s_{1b}r_1 \right] k_a - r_1 k_1 - \frac{1}{2(K^2 - s_{a1})} \left[(1-\rho)K^2 - 2s_{1a}r_1 \right] k_b\,, \\ k_{\hat b} &= -\frac{1}{2(K^2 - s_{1b})} \left[(1-\rho)K^2 - 2s_{1b}(1-r_1) \right] k_a - (1-r_1)k_1 \\ &\qquad - \frac{1}{2(K^2 - s_{a1})} \left[(1+\rho)K^2 - 2s_{1a}(1-r_1) \right] k_b\,, \end{aligned} \tag{5.3}$$

for arbitrary r_1, where $K = k_a + k_1 + k_b$ and

$$\rho = \sqrt{1 + \frac{4r_1(1-r_1)s_{1a}s_{1b}}{K^2 s_{ab}}}\,. \tag{5.4}$$

Note that $\rho \to 1$ in all singular limits, so these functions also satisfy the third requirement, of appropriate reduction in the different singular limits, so long as r_1 is not singular.

What about the last requirement, of avoiding spurious terms due to a momentum excess? The momentum excess in $J(a, 1, \hat a; k_b + k_{\hat b})$ is,

$$k_b+k_{\hat b} = (1-\rho)K^2 \left[\frac{k_b}{2(K^2 - s_{a1})} - \frac{k_a}{2(K^2 - s_{1b})} \right] + \frac{s_{1b}(1-r_1)}{K^2 - s_{1b}}\, k_a - (1-r_1)k_1 - \frac{s_{a1}r_1}{K^2 - s_{a1}}\, k_b\,, \tag{5.5}$$

Now, as $s_{a1} \to 0$, $\rho \sim 1+\mathcal{O}(s_{a1} r_1(1-r_1))$, so the terms proportional to $1-\rho$ satisfy the requirement even without taking into account any suppression from $r_1(1-r_1)$. The same is true of the very last term, explicitly proportional to s_{a1}. This leaves us with

$$(1-r_1)\Big[\frac{s_{1b}}{K^2-s_{1b}}\,k_a - k_1\Big] \tag{5.6}$$

for which the requirement will clearly be satisfied if $r_1 \sim 1+\mathcal{O}(s_{a1})$ in this limit, for example,

$$r_1 = \frac{s_{1b}}{s_{1a}+s_{1b}} \tag{5.7}$$

will do. (Given this requirement on r_1, ρ in fact goes like $1+\mathcal{O}(s_{a1}^2)$ in the limit. As I shall discuss in a later section, $r_1 \sim 1+\mathcal{O}(\sqrt{s_{a1}})$ would in fact be sufficient to ensure that eqn. (5.6) leads only to subleading contributions, though *not* for the seemingly obvious reason that the helicity algebra knocks down the $1/s_{a1}$ pole to a square-root singularity, because that doesn't happen in these additional terms.) With this choice, eqns. (5.3) are invariant under the simultaneous exchanges $\hat{a} \leftrightarrow \hat{b}, a \leftrightarrow b$, so that the other term in eqn. (4.5), involving the excess $k_a + k_{\hat{a}}$ in the limit $s_{1b} \to 0$, also avoids introducing additional or ambiguous terms into the antenna function. This also implies that we can ignore the momentum excess in the definition of the antenna amplitude, simplifying it to

$$\mathrm{Ant}(\hat{a},\hat{b} \leftarrow a,1,b) = J(a,1;\hat{a};0)J(b;\hat{b};0) + J(a;\hat{a};0)J(1,b;\hat{b};0)\,. \tag{5.8}$$

The choice (5.7) for the reconstruction functions is not unique, and other choices for r_1 are possible. Indeed, while the choice $r_1 = [s_{1b}/(s_{a1}+s_{1b})]^{1/17}$, for example, would fail to satisfy this scaling requirement as $s_{1b} \to 0$, the choice $r_1 = [s_{1b}/(s_{a1}+s_{1b})]^2$ would be satisfactory (though asymmetric).

The Catani–Seymour forms of the fused momenta for $k_a \parallel k_1$ can be obtained by setting $r_1 = 1$ in eqn. (5.3), while those for $k_b \parallel k_1$ can be obtained by setting $r_1 = 0$. Of course, these restricted forms are not appropriate in the opposite limits, while eqn. (5.3) interpolates smoothly between the two forms, approaching the Catani–Seymour forms in each of the two limits, as well as in the soft limit, $k_1 \to 0$.

6. General Reconstruction Function

In order to go beyond the emission of a lone gluon, we need to generalize the constructs of the previous section to handle more soft or collinear partons. We seek a pair of reconstruction functions, now a function of the $n+2$ momenta $k_a, k_1, \ldots, k_n, k_b$ with $k_{a,b}$ the surviving hard momenta. As before, we want to keep the reconstructed momenta $k_{\hat{a},\hat{b}}$ massless, and to conserve momentum,

$$K \equiv -(k_{\hat{a}} + k_{\hat{b}}) = k_a + k_1 + \ldots k_n + k_b\,. \tag{6.1}$$

We will also want the reconstruction functions to have the appropriate limits not only when all the numbered momenta become singular, but also in strongly-ordered limits, when a subset of these momenta become collinear with each other, or when a subset becomes much softer than other momenta. As for the single-emission case, there will be scaling constraints arising from the need to leave the leading singularity in the antenna amplitude unchanged; I defer a discussion of them until the next section.

The constraints of masslessness and momentum conservation can be satisfied by the following functional forms,

$$\begin{aligned} k_{\hat a} &= -\frac{1}{2(K^2-t_{1\cdots nb})}\left[(1+\rho)K^2 + 2R\cdot(k_a-k_b-K) + \frac{1}{s_{ab}}G\begin{pmatrix} k_a, k_b \\ R, K_{1,n}\end{pmatrix}\right] k_a - R \\ &\quad - \frac{1}{2(K^2-t_{a1\cdots n})}\left[(1-\rho)K^2 + 2R\cdot(k_b-k_a-K) + \frac{1}{s_{ab}}G\begin{pmatrix} k_b, k_a \\ R, K_{1,n}\end{pmatrix}\right] k_b \\ k_{\hat b} &= -\frac{1}{2(K^2-t_{1\cdots nb})}\left[(1-\rho)K^2 + 2\widetilde{R}\cdot(k_a-k_b-K) + \frac{1}{s_{ab}}G\begin{pmatrix} k_a, k_b \\ \widetilde{R}, K_{1,n}\end{pmatrix}\right] k_a - \widetilde{R} \\ &\quad - \frac{1}{2(K^2-t_{a1\cdots n})}\left[(1+\rho)K^2 + 2\widetilde{R}\cdot(k_b-k_a-K) + \frac{1}{s_{ab}}G\begin{pmatrix} k_b, k_a \\ \widetilde{R}, K_{1,n}\end{pmatrix}\right] k_b \end{aligned} \tag{6.2}$$

where $t_{1\cdots nb} = (k_1+\cdots+k_n+k_b)^2$ (so that $K^2-t_{1\cdots nb} = 2k_a\cdot(K_{1,n}+k_b)$); $t_{a1\cdots n} = (k_a+k_1+\cdots+k_n)^2$;

$$R = \sum_{j=1}^{n} k_j r_j\,, \qquad \widetilde{R} = K_{1,n} - R = \sum_{j=1}^{n} k_j(1-r_j)\,; \tag{6.3}$$

and

$$\rho = \left[1 + \frac{2\,G\begin{pmatrix} a,R,b \\ a,\widetilde{R},b\end{pmatrix}}{K^2 s_{ab}^2} + \frac{\Delta(a,R,K,b)}{(K^2)^2 s_{ab}^2}\right]^{1/2}. \tag{6.4}$$

Let us examine the limits of these functions when various combinations of momenta become collinear or soft. If two adjacent numbered legs, say j and $j{+}1$, become collinear, then the reconstruction functions for $(k_a, k_1, \ldots, k_j, k_{j+1}, \ldots, k_n, k_b)$ change smoothly into those for $(k_a, k_1, \ldots, k_j+k_{j+1}, \ldots, k_n, k_b)$ so long as $r_{j+1} \to r_j$ (and none of the r_i are singular). Similarly, the reconstruction functions reduce smoothly when k_j becomes soft, so long as r_j is not singular in the limit.

In the limit when k_1 becomes collinear to k_a, define $k_A \equiv k_a + k_1$, and take z_a and z_1 to be the momentum fractions of k_a and k_1 with respect to k_A. Let primed variables represent sums omitting k_1, e.g. R'; then

$$\rho \to \rho' = \left[1 + \frac{2\,G\begin{pmatrix} A,R',b \\ A,\widetilde{R}',b \end{pmatrix}}{K^2 s_{Ab}^2} + \frac{\Delta(A,R',K,b)}{(K^2)^2 s_{Ab}^2}\right]^{1/2} .$$

$$\begin{aligned}
k_{\hat a} \to &-\frac{1}{2z_a(K^2-t_{2\cdots nb})}\Bigg[(1+\rho')K^2 + 2R'\cdot(k_A-k_b-K) + \frac{1}{s_{Ab}}G\begin{pmatrix} k_A,k_b \\ R',K_{2,n} \end{pmatrix} \\
&\qquad - 2z_1r_1k_A\cdot(k_b+K) - 2z_1R'\cdot k_A + \frac{z_1r_1}{s_{Ab}}G\begin{pmatrix} k_A,k_b \\ k_A,K_{2,n} \end{pmatrix} + \frac{z_1}{s_{Ab}}G\begin{pmatrix} k_A,k_b \\ R',k_A \end{pmatrix}\Bigg](z_ak_A) \\
&- z_1r_1k_A - R' - \frac{1}{2(K^2-t_{A2\cdots n})}\Bigg[(1-\rho')K^2 + 2R'\cdot(k_b-k_A-K) + \frac{1}{s_{Ab}}G\begin{pmatrix} k_b,k_A \\ R',K_{2,n} \end{pmatrix} \\
&\qquad + 2z_1r_1k_A\cdot(k_b-K) + 2z_1R'\cdot k_A + \frac{z_1r_1}{s_{Ab}}G\begin{pmatrix} k_b,k_A \\ k_A,K_{2,n} \end{pmatrix} + \frac{z_1}{s_{Ab}}G\begin{pmatrix} k_b,k_A \\ R',k_A \end{pmatrix}\Bigg]k_b \\
= &-\frac{1}{2(K^2-t_{2\cdots nb})}\Bigg[(1+\rho')K^2 + 2R'\cdot(k_A-k_b-K) + \frac{1}{s_{Ab}}G\begin{pmatrix} k_A,k_b \\ R',K_{2,n} \end{pmatrix}\Bigg]k_A - R' \\
&+ \frac{z_1r_1k_A\cdot(k_b+K+K_{2,n})}{(K^2-t_{2\cdots nb})}k_A - z_1r_1k_A - \frac{z_1k_A\cdot(k_b-K+K_{2,n})}{2(K^2-t_{A2\cdots n})}r_1k_b \\
&- \frac{1}{2(K^2-t_{A2\cdots n})}\Bigg[(1-\rho')K^2 + 2R'\cdot(k_b-k_A-K) + \frac{1}{s_{Ab}}G\begin{pmatrix} k_b,k_A \\ R',K_{2,n} \end{pmatrix}\Bigg]k_b\,;
\end{aligned} \tag{6.5}$$

the terms proportional to r_1 on the penultimate line cancel, leaving exactly the form required for the reconstruction function from $(k_A, k_2, \ldots, k_n, k_b)$. The derivation for $k_{\hat b}$ is similar. Because there are no singularities, this generalizes to a collection of momenta $\{k_j, \ldots, k_{j+l}\}$ becoming collinear. The reconstruction functions thus have the correct form in any strongly-ordered limit.

The general limit in which we are interested involves more than one momentum becoming singular: a subset of momenta in $\{k_1, \ldots, k_j\}$ will become collinear with k_a; a subset of momenta in $\{k_{j+1}, \ldots, k_n\}$ will become collinear with k_b; and all remaining momenta will become soft†. The soft momenta disappear quietly from the expressions for $k_{\hat a,\hat b}$ (so long as none of the r_i are singular), and $\rho \to 1$. Define

$$\begin{aligned}
k_A &\equiv k_a + k_1 + \cdots + k_j\,, \\
k_B &\equiv k_{j+1} + \cdots + k_n + k_b\,, \\
z_{l,m} &\equiv \sum_{j=l}^{m} z_j\,, \\
\hat r_{l,m} &\equiv \sum_{j=l}^{m} r_j z_j\,,
\end{aligned} \tag{6.6}$$

† As explained by Campbell and Glover [16], it is sufficient to examine the limits of amplitudes for configurations where the collinear momenta within each of the two sets are color-connected.

where $z_1 \cdots z_j$ and $z_{j+1} \cdots z_n$ refer to momentum fractions with respect to k_A and k_B respectively. We have the limit

$$
\begin{aligned}
k_{\hat{a}} \to & -\frac{1}{2s_{AB}} \Bigg[2K^2 - \hat{r}_{j+1,n} z_{1,j} s_{AB} - \hat{r}_{1,j}(z_b+1) s_{AB} \\
& \qquad + \frac{\hat{r}_{1,j} z_{j+1,n}}{s_{AB}} G\begin{pmatrix} k_A, k_B \\ k_A, k_B \end{pmatrix} + \frac{\hat{r}_{j+1,n} z_{1,j}}{s_{AB}} G\begin{pmatrix} k_A, k_B \\ k_B, k_A \end{pmatrix} \Bigg] k_A - \hat{r}_{1,j} k_A - \hat{r}_{j+1,n} k_B \\
& - \frac{1}{2s_{AB}} \Bigg[2K^2 - \hat{r}_{1,j} z_{j+1,n} s_{AB} - \hat{r}_{j+1,n}(z_a+1) s_{AB} \\
& \qquad + \frac{\hat{r}_{1,j} z_{j+1,n}}{s_{AB}} G\begin{pmatrix} k_B, k_A \\ k_A, k_B \end{pmatrix} + \frac{\hat{r}_{j+1,n} z_{1,j}}{s_{AB}} G\begin{pmatrix} k_B, k_A \\ k_B, k_A \end{pmatrix} \Bigg] k_B \\
= & -[1-\hat{r}_{1,j}]\, k_A - \hat{r}_{1,j} k_A - \hat{r}_{j+1,n} k_B - [-\hat{r}_{j+1,n}]\, k_B \\
= & -k_A \,,
\end{aligned}
\tag{6.7}
$$

as desired.

For later purposes, we will also need the leading corrections to the Gram determinants in the approach to the limit. For this purpose, introduce two small parameters, to scale the invariants involving a and b respectively,

$$
\begin{aligned}
s_{a1}, t_{a12}, \ldots, t_{a1\cdots j} &\propto \delta_a \,, \\
s_{nb}, \ldots, t_{(j+1)\cdots nb} &\propto \delta_b \,,
\end{aligned}
\tag{6.8}
$$

though of course the two must be of the same order.

It is clear that expressions like $G\left(\begin{smallmatrix} a, K_{1,j}, b \\ a, K_{j+1,n}, b \end{smallmatrix}\right)$ vanish in the limit; but how quickly do they do so? To understand this, first examine a simpler vanishing object in the limit $k_a \parallel k_1$,

$$
G\begin{pmatrix} a, 1, b \\ a, q, b \end{pmatrix} = s_{ab}(s_{1b}s_{aq} - s_{1q}s_{ab} + s_{1a}s_{bq})
\tag{6.9}
$$

with q an arbitrary null vector. The last term inside the parentheses is of $\mathcal{O}(\delta_a)$, and the first two terms clearly cancel in the limit. To understand the size of the corrections, it is convenient to rewrite them in terms of spinor products,

$$
s_{1b}s_{aq} - s_{1q}s_{ab} = \langle 1\, b\rangle \langle a\, q\rangle [1\, b] [a\, q] - \langle 1\, q\rangle \langle a\, b\rangle [1\, q] [a\, b] \,,
\tag{6.10}
$$

and to apply the Schouten identity several times,

$$
s_{1b}s_{aq} - s_{1q}s_{ab} = \langle 1\, a\rangle \langle b\, q\rangle [1\, b] [a\, q] + [1\, a] [b\, q] \langle 1\, b\rangle \langle a\, q\rangle - s_{1a}s_{bq} \,,
\tag{6.11}
$$

so that overall the expression in eqn. (6.9) is manifestly of $\mathcal{O}(\sqrt{\delta_a})$ in the limit. Each collinear pair will clearly contribute a factor of $\sqrt{\delta}$, so that

$$\begin{aligned}
G\begin{pmatrix} K_{1,j}, a \\ q, b \end{pmatrix} &\sim \mathcal{O}(\sqrt{\delta_a})\,, \\
G\begin{pmatrix} K_{j+1,n}, b \\ q, a \end{pmatrix} &\sim \mathcal{O}(\sqrt{\delta_b})\,, \\
G\begin{pmatrix} a, K_{1,j}, b \\ a, K_{j+1,n}, b \end{pmatrix} &\sim \mathcal{O}(\sqrt{\delta_a \delta_b})\,, \\
G\begin{pmatrix} a, K_{1,j}, K_{j+1,n}, b \\ a, K_{1,j}, K_{j+1,n}, b \end{pmatrix} &\sim \mathcal{O}(\delta_a \delta_b)\,.
\end{aligned} \tag{6.12}$$

7. Antenna Amplitude for Double Emission

I turn next to the construction of the antenna amplitude itself. In this section, I will consider the emission of two singular gluons, deferring the derivation of a general form to the next section. We are interested in the following singular configurations,

(a) $t_{a12}, s_{a1}, s_{12} \longrightarrow 0$, with t_{12b} and s_{2b} approaching different constant limits, that is the multiply-collinear limit $k_{1,2} \parallel k_a$;

(b) $s_{a1}, s_{2b} \longrightarrow 0$, with t_{a12}, t_{12b}, and s_{12} approaching different constant limits, that is the double collinear limit $k_1 \parallel k_a$ and $k_2 \parallel k_b$;

(c) $t_{12b}, s_{2b}, s_{12} \longrightarrow 0$, with t_{a12} and s_{a1} approaching different constant limits, that is the multiply-collinear limit $k_{1,2} \parallel k_b$;

(d) $t_{a12}, s_{a1}, s_{12}, s_{2b} \longrightarrow 0$, with t_{12b} approaching a constant limit, that is the collinear-soft limit $k_1 \parallel k_a$ and k_2 soft;

(e) $t_{12b}, s_{2b}, s_{12}, s_{a1} \longrightarrow 0$, with t_{a12} approaching a constant limit, that is the collinear-soft limit $k_2 \parallel k_b$ and k_1 soft;

(f) All invariants $t_{a12}, s_{a1}, s_{12}, s_{2b}, t_{12b} \longrightarrow 0$, that is the double-soft limit where $k_{1,2}$ are both soft. This is the only case where one of the invariants (s_{12}) is much smaller than the others.

In all cases, s_{ab} again defines a hard scale, and the non-collinear momentum in each region acts as a reference momentum to define the collinear momentum fractions. (As pointed out by Campbell and Glover [16], we need not concern ourselves with the limit $k_2 \parallel k_a$ where k_1 is soft, nor with the limit $k_1 \parallel k_b$ where k_2 is soft, because the amplitude, while singular in these limits, will not be sufficiently so to yield any poles in $(D-4)$ from integrating over the phase space of singular emission.)

In general, none of the small invariants are significantly smaller than the others, though strongly-ordered limits are included as degenerate cases. In all cases, however, the two-particle invariants will vanish at least as quickly as the three-particle invariants: s_{a1}/t_{a12}, for example, will be bounded above by a constant. (If all particles are in the final state, that constant is 1; if some are in the initial state, it can be larger than 1, but in any event we are *not* dealing with configurations where $t_{a12} \to 0$ without the two-particle invariants s_{a1}, s_{12} getting small.) We want to extract all terms that scale as $1/\delta$ or more singular when the invariant shrinks with δ, dropping less singular terms. Equivalently, we are interested in terms singular when all of the ratios $\Delta(a,1,2,b)/(\Delta(a,1,b)s_{ab}), \Delta(a,1,2,b)/(\Delta(a,2,b)s_{ab}), \Delta(a,1,b)/s_{ab}^3$, and $\Delta(a,2,b)/s_{ab}^3$ tend to zero.

Begin by extracting all terms in the n-point amplitude which have singularities in either t_{a12}; s_{a1} and s_{2b}; or t_{12b}. As explained in section 3, if we work in light-cone gauge, then all poles in s_{a1} or s_{2b} in the regions where one of the three-particle invariants vanishes will be contained in the first or last of these contributions. This gives us the following form,

$$\begin{aligned}
&J(a,1,2;-(k_a+k_1+k_2);0)A_{n-2}(\ldots,k_a+k_1+k_2,b,\ldots)\\
&+J(a,1;-(k_a+k_1);0)J(2,b;-(k_2+k_b);0)A_{n-2}(\ldots,k_a+k_1,k_b+k_2,\ldots)\\
&+J(1,2,b;-(k_1+k_2+k_b);0)A_{n-2}(\ldots,a,k_b+k_1+k_2,\ldots)
\end{aligned}\tag{7.1}$$

I do *not* extract a contribution proportional to $J(1,2;\cdot;\cdot)$; while such a term indeed gives rise to an s_{12} pole, it cannot give rise to singularities in other invariants, and hence would yield only a subleading contribution. That is, if we examine those diagrams containing only three-point vertices, and ignore the helicity algebra which will soften the poles, we need to extract all terms that have poles in two invariants; one invariant will not suffice.

As in the single-emission case, introduce a complete set of polarization states for the unfused leg in the first and last terms, rewriting the product of polarization vectors as a two-point current; and introduce new labels for the surviving hard momenta, yielding

$$\begin{aligned}
&J(a,1,2;-(k_a+k_1+k_2);0)J(b;-k_b;0)A_{n-2}(\ldots,-k_{\hat a}=k_a+k_1+k_2,-k_{\hat b}=k_b,\ldots)\\
&+J(a,1;-(k_a+k_1);0)J(2,b;-(k_2+k_b);0)A_{n-2}(\ldots,-k_{\hat a}=k_a+k_1,-k_{\hat b}=k_b+k_2,\ldots)\\
&+J(a;-k_a;0)J(1,2,b;-(k_1+k_2+k_b);0)A_{n-2}(\ldots,-k_{\hat a}=k_a,-k_{\hat b}=k_b+k_1+k_2,\ldots)
\end{aligned}\tag{7.2}$$

Using the reconstruction functions defined in the previous section, and again putting the hatted momenta on-shell (at the price of allowing momentum to flow between the two currents in each term), we can combine terms to obtain an antenna factorization amplitude,

$$\begin{aligned}
\mathrm{Ant}(\hat a,\hat b\leftarrow a,1,2,b) = &J(a,1,2;\hat a;k_b+k_{\hat b})J(b;\hat b;k_a+k_1+k_2+k_{\hat a})\\
&+J(a,1;\hat a;k_b+k_2+k_{\hat b})J(2,b;\hat b;k_a+k_1+k_{\hat a})\\
&+J(a;\hat a;k_b+k_1+k_2+k_{\hat b})J(1,2,b;\hat b;k_a+k_{\hat a})\,,
\end{aligned}\tag{7.3}$$

and corresponding factorization in any singular limit,

$$\mathrm{Ant}(\hat a,\hat b\leftarrow a,1,2,b)A_{n-2}(\ldots,-k_{\hat a}=f_{\hat a}(k_a,k_1,k_2,k_b),-k_{\hat b}=f_{\hat b}(k_a,k_1,k_2,k_b),\ldots)\,. \tag{7.4}$$

In this equation, the summation over physical polarizations of $\hat{a}$ and $\hat{b}$ is implicit.

We must still impose the requirement that the momentum excess in each current leads only to subleading terms. This will lead to constraints on the coefficients r_i. In the last term of eqn. (7.3), we must examine the behavior of $k_a+k_{\hat{a}}$ as $k_{1,2}$ become collinear to k_b (the argument will also hold when either or both become soft),

$$\begin{aligned}
k_a+k_{\hat{a}} \sim \Bigg[\frac{1-\rho}{2} + \frac{t_{12b}}{4k_a\cdot(k_b+K_{1,2})\,s_{ab}}\bigg\{(1-\rho)s_{ab} - 2s_{ab} + s_{a2}\frac{s_{1b}}{t_{12b}}(r_1-r_2) - s_{a1}\frac{s_{2b}}{t_{12b}}(r_1-r_2) \\
+ s_{ab}\frac{s_{12}}{t_{12b}}(r_1+r_2) + 2s_{ab}\frac{s_{1b}}{t_{12b}}r_1 + 2s_{ab}\frac{s_{2b}}{t_{12b}}r_2\bigg\}\Bigg] k_a \\
- r_1k_1 - r_2k_2 \\
+ \frac{1}{2s_{ab}}\Bigg[2(\rho-1)\,k_a\cdot(k_b+K_{1,2}) + 2s_{a1}r_1 + 2s_{a2}r_2 \\
+ \frac{t_{12b}}{s_{ab}}\bigg\{(1-\rho)(s_{a1}+s_{a2}) + 2(\rho-1)\frac{s_{12}}{t_{12b}}k_a\cdot(k_b+K_{1,2}) - s_{a1}\frac{s_{2b}}{t_{12b}}(r_1+r_2) \\
- s_{a2}\frac{s_{1b}}{t_{12b}}(r_1+r_2) + s_{ab}\frac{s_{12}}{t_{12b}}(r_1+r_2) - 2s_{a1}\frac{s_{1b}}{t_{12b}}r_1 - 2s_{a2}\frac{s_{2b}}{t_{12b}}r_1\bigg\}\Bigg] k_b \\
+ \mathcal{O}(t_{12b}^2).
\end{aligned} \tag{7.5}$$

Also,

$$\begin{aligned}
\rho \sim 1 + \frac{t_{12b}}{2k_a\cdot(k_b+K_{1,2})\,s_{ab}}\bigg\{2s_{a1}r_1(1-r_1)\frac{s_{1b}}{t_{12b}} + 2s_{a2}r_2(1-r_2)\frac{s_{2b}}{t_{12b}} \\
+ (r_1+r_2-2r_1r_2)\left(s_{a2}\frac{s_{1b}}{t_{12b}} + s_{a1}\frac{s_{2b}}{t_{12b}} - s_{ab}\frac{s_{12}}{t_{12b}}\right)\bigg\} + \mathcal{O}(t_{12b}^2)\,.
\end{aligned} \tag{7.6}$$

As noted above, for the limits in which we are interested, the ratios s_{1b}/t_{12b}, s_{2b}/t_{12b}, and s_{12}/t_{12b} are all bounded in regions where $t_{12b}\to 0$. Most of the terms in $k_a+k_{\hat{a}}$ therefore kill off the leading t_{12b} pole in $J(1,2,b;\hat{b};\cdot)$, and leave only subleading contributions. The surviving terms are,

$$-r_1k_1 - r_2k_2 + \frac{s_{a1}}{s_{ab}}r_1k_b + \frac{s_{a2}}{s_{ab}}r_2k_b\,. \tag{7.7}$$

The choice

$$r_j = \frac{k_j\cdot(K_{j+1,n}+k_b)}{k_j\cdot K} = \frac{t_{j\cdots nb} - t_{(j+1)\cdots nb}}{2k_j\cdot K} \tag{7.8}$$

ensures that these vanish like a full power of a small invariant as as $t_{12b}\to 0$,

$$\begin{aligned}
r_1 &= \frac{t_{12b}}{2k_1\cdot K}\left(1-\frac{s_{2b}}{t_{12b}}\right)\,, \\
r_2 &= \frac{t_{12b}}{2k_2\cdot K}\frac{s_{2b}}{t_{12b}}\,,
\end{aligned} \tag{7.9}$$

so that the remaining terms also kill off the leading pole, leaving only subleading contributions to the antenna amplitude. In fact, a closer examination of the terms in eqn. (7.7) shows that a square-root vanishing would have been sufficient: we recognize the dot product of that vector with any other vector w as

$$-\frac{r_1}{s_{ab}} G\begin{pmatrix} k_b, k_1 \\ k_a, w \end{pmatrix} - \frac{r_2}{s_{ab}} G\begin{pmatrix} k_b, k_2 \\ k_a, w \end{pmatrix}, \tag{7.10}$$

which, as discussed in the previous section, already vanishes like the square root of a small invariant in the limit. Similarly, in the single-emission case, taking $r_1 \sim \sqrt{s_{a1}}$ in the limit would have been sufficient to avoid changing the leading singularity.

A similar argument holds for the first term in eqn. (7.3), and the choice of r_i in eqn. (7.8) will lead to contributions of $\mathcal{O}(t_{a12})$ from the excess momentum, that is, subleading contributions.

In the middle term of eqn. (7.3), we need to examine the behavior of $k_a + k_{\hat{a}} + k_1$ and $k_b + k_{\hat{b}} + k_2$ as k_1 becomes collinear to k_a, and k_2 to k_b,

$$\begin{aligned}
&k_{\hat{a}} + k_a + k_1 \sim \\
&\quad \frac{1}{2s_{ab}(s_{a2}+s_{ab})}\Bigg[(s_{1b}s_{a2} - s_{12}s_{ab})\left(1 - \frac{s_{a1}}{s_{a2}+s_{ab}}\right) + (1-\rho)s_{ab}(s_{12}+s_{1b})\left(1 - \frac{s_{a1}}{s_{a2}+s_{ab}}\right) \\
&\qquad\qquad + (1-\rho)s_{ab}(s_{a2}+s_{ab})\left(1 + \frac{s_{2b}}{s_{a2}+s_{ab}}\right) \\
&\qquad\qquad + (r_1 - 1)\left(1 - \frac{s_{a1}}{s_{a2}+s_{ab}}\right)(s_{1b}s_{a2} + s_{12}s_{ab} + 2s_{1b}s_{ab}) \\
&\qquad\qquad + r_2(s_{12}s_{ab} - s_{1b}s_{a2})\left(1 - \frac{s_{a1}}{s_{a2}+s_{ab}}\right) + 2(r_2-1)s_{2b}s_{ab}\Bigg]k_a + (1-r_1)k_1 \\
&\quad + \frac{1}{2s_{ab}(s_{1b}+s_{ab})}\Bigg[(s_{12}s_{ab} - s_{1b}s_{a2})\left(1 - \frac{s_{2b}}{s_{1b}+s_{ab}}\right) + (\rho-1)s_{ab}(s_{12}+s_{a2})\left(1 - \frac{s_{2b}}{s_{1b}+s_{ab}}\right) \\
&\qquad\qquad + (\rho-1)s_{ab}(s_{1b}+s_{ab})\left(1 + \frac{s_{a1}}{s_{1b}+s_{ab}}\right) \\
&\qquad\qquad + (r_1-1)(s_{12}s_{ab} - s_{1b}s_{a2})\left(1 - \frac{s_{2b}}{s_{1b}+s_{ab}}\right) + 2r_1 s_{a1}s_{ab} \\
&\qquad\qquad + r_2(s_{1b}s_{a2} + s_{12}s_{ab} + 2s_{a2}s_{ab})\left(1 - \frac{s_{2b}}{s_{1b}+s_{ab}}\right)\Bigg]k_b - r_2 k_2\,,
\end{aligned} \tag{7.11}$$

$$
\begin{aligned}
&k_{\hat b}+k_b+k_2 \sim\\
&\quad\frac{1}{2s_{ab}(s_{2a}+s_{ab})}\Bigg[(s_{12}s_{ab}-s_{1b}s_{a2})\left(1-\frac{s_{a1}}{s_{a2}+s_{ab}}\right)+(\rho-1)(s_{12}+s_{1b})\left(1-\frac{s_{a1}}{s_{a2}+s_{ab}}\right)\\
&\qquad\qquad+(\rho-1)(s_{a2}+s_{ab})\left(1+\frac{s_{2b}}{s_{a2}+s_{ab}}\right)\\
&\qquad\qquad+(1-r_1)(s_{1b}s_{a2}+s_{12}s_{ab}+2s_{1b}s_{ab})\left(1-\frac{s_{a1}}{s_{a2}+s_{ab}}\right)\\
&\qquad\qquad+2(1-r_2)s_{2b}s_{ab}-r_2(s_{12}s_{ab}-s_{1b}s_{a2})\left(1-\frac{s_{a1}}{s_{a2}+s_{ab}}\right)\Bigg]k_a-(1-r_1)k_1\\
&\quad+\frac{1}{2s_{ab}(s_{1b}+s_{ab})}\Bigg[(s_{1b}s_{a2}-s_{12}s_{ab})\left(1-\frac{s_{2b}}{s_{1b}+s_{ab}}\right)+(1-\rho)s_{ab}(s_{12}+s_{a2})\left(1-\frac{s_{2b}}{s_{1b}+s_{ab}}\right)\\
&\qquad\qquad+(1-\rho)s_{ab}(s_{1b}+s_{ab})\left(1+\frac{s_{a1}}{s_{1b}+s_{ab}}\right)\\
&\qquad\qquad+(r_1-1)(s_{12}s_{ab}-s_{1b}s_{a2})\left(1-\frac{s_{2b}}{s_{1b}+s_{ab}}\right)-2r_1s_{a1}s_{ab}\\
&\qquad\qquad-r_2\left(1-\frac{s_{2b}}{s_{1b}+s_{ab}}\right)(s_{1b}s_{a2}+s_{12}s_{ab}+2s_{a2}s_{ab})\Bigg]k_b+r_2k_2.
\end{aligned}
\tag{7.12}
$$

Any term proportional to *either* s_{a1} or s_{2b} will kill off the pole in $J(a,1;\hat a;k_b+k_2+k_{\hat b})$ or $J(2,b;\hat b;k_a+k_1+k_{\hat a})$ respectively, giving rise only to subleading contributions. Also, using eqn. (7.8), we find

$$
\begin{aligned}
r_1-1&=-\frac{k_1\cdot(k_a+k_1)}{k_1\cdot K}=\frac{s_{a1}}{2k_1\cdot K}\,,\\
r_2&=\frac{s_{2b}}{2k_2\cdot K}\,,
\end{aligned}
\tag{7.13}
$$

so terms proportional to (r_1-1) or r_2 can be dropped as well. This leaves us with,

$$
\begin{aligned}
&k_{\hat a}+k_a+k_1\sim\\
&\quad\frac{1}{2s_{ab}(s_{a2}+s_{ab})}\Big[(s_{1b}s_{a2}-s_{12}s_{ab})+(1-\rho)s_{ab}(s_{12}+s_{1b}+s_{a2}+s_{ab})\Big]k_a\\
&\quad+\frac{1}{2s_{ab}(s_{1b}+s_{ab})}\Big[(s_{12}s_{ab}-s_{1b}s_{a2})+(\rho-1)s_{ab}(s_{12}+s_{a2}+s_{1b}+s_{ab})\Big]k_b\,,\\
&k_{\hat b}+k_b+k_2\sim\\
&\quad\frac{1}{2s_{ab}(s_{2a}+s_{ab})}\Big[(s_{12}s_{ab}-s_{1b}s_{a2})+(\rho-1)(s_{12}+s_{1b}+s_{a2}+s_{ab})\Big]k_a\\
&\quad+\frac{1}{2s_{ab}(s_{1b}+s_{ab})}\Big[(s_{1b}s_{a2}-s_{12}s_{ab})+(1-\rho)s_{ab}(s_{12}+s_{a2}+s_{1b}+s_{ab})\Big]k_b\,.
\end{aligned}
\tag{7.14}
$$

We recognize $(s_{1b}s_{a2}-s_{12}s_{ab})$ as $G\left({a,1,b \atop a,2,b}\right)/s_{ab}$, which as discussed in the previous section, is $\sim\sqrt{s_{a1}s_{2b}}$ in the limit; likewise,

$$
\rho-1\sim\frac{(s_{12}s_{ab}-s_{a2}s_{1b})(s_{1b}s_{2a}+s_{12}s_{ab}+2(s_{2a}+s_{1b})s_{ab}+2s_{ab}^2)}{2s_{ab}^2(s_{a2}+s_{12}+s_{1b}+s_{ab})^2}\sim\sqrt{s_{a1}s_{2b}}\,,
\tag{7.15}
$$

so that the surviving terms in both $k_{\hat{a}} + k_a + k_1$ and $k_{\hat{b}} + k_b + k_2$ are all of $\mathcal{O}(\sqrt{s_{a1}s_{2b}})$.

Next, split $J(a,1;\hat{a};k_b{+}k_2{+}k_{\hat{b}})$ and $J(2,b;\hat{b};k_a{+}k_1{+}k_{\hat{a}})$ each into a 'canonical' term and an 'excess' term. The former is defined as the current with the excess momenta set to zero, while the latter contains the excess momenta,

$$\begin{aligned} J^{\text{canon}}(a,1;\hat{a};k_b{+}k_2{+}k_{\hat{b}}) &= J(a,1;\hat{a};0)\,, \\ J^{\text{excess}}(a,1;\hat{a};k_b{+}k_2{+}k_{\hat{b}}) &= J(a,1;\hat{a};k_b{+}k_2{+}k_{\hat{b}}) - J(a,1;\hat{a};0)\,. \end{aligned} \tag{7.16}$$

Because the off-shell momentum does not appear on the right-hand side of the recurrence relation (2.1), the canonical term is in fact the same as the current with the hatted argument replaced by the off-shell momentum, respectively $-(k_a+k_1)$ or $-(k_2+k_b)$ in the two currents. In the canonical terms, while an s_{a1} or s_{2b} pole appears in the formal expression for the currents, evaluation of the polarization vectors will soften the singularity to a square-root one. In the excess terms, no such softening will necessarily occur. The product of the two canonical terms gives us a term in the antenna amplitude; we want to show that either the product of a canonical and an excess term, or the product of the two excess terms, yield only subleading terms in this singular region. Since each excess term is $\sim \sqrt{s_{a1}s_{2b}}$ in the limit, the product indeed kills off both poles, as desired. In the product of an excess term and a canonical term, the strength of the canonical term's pole will be reduced only by a square root of the pole invariant; but here, this suffices because the canonical term only has a square-root singularity rather than a full pole in the invariant.

Thus, in each of the regions where a given term in the antenna amplitude contributions, the excess momentum transferred between the currents gives rise to no corrections to the leading singularity. We can thus use a simplified definition, ignoring this excess momentum,

$$\begin{aligned} \mathrm{Ant}(\hat{a},\hat{b} \leftarrow a,1,2,b) =& J(a,1,2;\hat{a};0)J(b;\hat{b};0) \\ &+ J(a,1;\hat{a};0)J(2,b;\hat{b};0) \\ &+ J(a;\hat{a};0)J(1,2,b;\hat{b};0)\,. \end{aligned} \tag{7.17}$$

8. Antenna Amplitude for Multiple Emission

The construction and arguments of the previous section generalize straightforwardly to the case of multiple singular emission. In the general case, with emission of m singular gluons, we are interested in all terms with singularities in at least m invariants. To extract these, we isolate all terms with poles in $t_{a1\cdots m}$; $t_{a1\cdots(m-1)}$ and s_{mb}; $t_{a1\cdots(m-2)}$ and $t_{(m-1)mb}$; and so on through terms with poles in $t_{1\cdots mb}$. As in the double-emission case, these terms will necessarily have singularities in additional variables. For example the term isolated via simultaneous poles in $t_{a1\cdots j}$ and $t_{(j+1)\cdots mb}$

will also contain poles in $s_{a1}, t_{a12}, \ldots, t_{a1\cdots(j-1)}$. This extraction yields the following form,

$$\begin{aligned}
&J(a,1,\ldots,m;-(k_a+K_{1,m});0)A_{n-m}(\ldots,k_a+K_{1,m},b,\ldots)\\
&\quad+\sum_{j=1}^{m-1} J(a,1,\ldots,j;-(k_a+K_{1,j});0)J(j+1,\ldots,m,b;-(K_{j+1,m}+k_b);0)\\
&\qquad\qquad\times A_{n-m}(\ldots,k_a+K_{1,j},k_b+K_{j+1,m},\ldots)\\
&\quad+J(1,\ldots,m,b;-(k_1+\cdots+k_m+k_b);0)A_{n-m}(\ldots,a,k_b+K_{1,m},\ldots)
\end{aligned} \tag{8.1}$$

As in the single- and double-emission cases, introduce a complete set of polarization states for the unfused leg in the first and last terms, rewriting the product of polarization vectors as a two-point current. Again introducing new labels for the surviving hard momenta, we obtain for the factorization,

$$\begin{aligned}
&\sum_{j=0}^{m} J(a,1,\ldots,j;-(k_a+K_{1,j});0)J(j+1,\ldots,m,b;-(k_b+K_{j+1,m});0)\\
&\qquad\qquad\times A_{n-m}(\ldots,-k_{\hat a}=k_a+K_{1,j},-k_{\hat b}=k_b+K_{j+1,m},\ldots)
\end{aligned} \tag{8.2}$$

Using the reconstruction functions defined in section 6, and again putting the hatted momenta on-shell (at the price of allowing momentum to flow between the two currents in each term), we can combine terms to obtain a general antenna factorization amplitude,

$$\begin{aligned}
&\mathrm{Ant}(\hat a,\hat b\leftarrow a,1,\ldots,m,b)=\\
&\qquad\sum_{j=0}^{m} J(a,1,\ldots,j;\hat a;k_b+k_{\hat b}+K_{j+1,m})J(j+1,\ldots,m,b;\hat b;k_a+k_{\hat a}+K_{1,j})
\end{aligned} \tag{8.3}$$

and corresponding factorization in singular limits,

$$\mathrm{Ant}(\hat a,\hat b\leftarrow a,1,\ldots,m,b)A_{n-m}(\ldots,-k_{\hat a}=f_{\hat a}(k_a,k_1,\ldots,k_m,k_b),-k_{\hat b}=f_{\hat b}(k_a,k_1,\ldots,k_m,k_b),\ldots)\,, \tag{8.4}$$

where again the summation over physical polarizations of $\hat a$ and $\hat b$ is implicit.

We can now verify that the choice (7.8) for the coefficients r_i ensures that the momentum excesses in this equation lead only to subleading terms. In a generic term, $0<j<m$, we must examine the behavior of $k_a+k_{\hat a}+k_1+\cdots+k_j$ as $k_{j+1,\ldots,m}$ become collinear to k_b. In terms of the small parameters $\delta_{a,b}$ introduced in eqn. (6.8), the coefficients $r_i\sim 1+\mathcal{O}(\delta_a)$ for $1\le i\le j$ and $r_i\sim\mathcal{O}(\delta_b)$ for $j+1\le i\le n$. Define δr_i via

$$r_i=\begin{cases} 1+\delta r_i\,, & 1\le i\le j;\\ \delta r_i\,, & j+1\le i\le n;\end{cases} \tag{8.5}$$

and $\delta R_{l,m}=\Sigma_{j=l}^{m}\delta r_j k_j$.

We then see that

$$
\begin{aligned}
\rho^2 \sim 1 + \frac{2}{K^2 s_{ab}^2} \Bigg[& G\begin{pmatrix} a, K_{1,j}, b \\ a, K_{j+1,n}, b \end{pmatrix} - G\begin{pmatrix} a, K_{1,j}, b \\ a, \delta R_{1,j}, b \end{pmatrix} - G\begin{pmatrix} a, K_{1,j}, b \\ a, \delta R_{j+1,n}, b \end{pmatrix} \\
& +G\begin{pmatrix} a, \delta R_{1,j}, b \\ a, K_{j+1,n}, b \end{pmatrix} + G\begin{pmatrix} a, \delta R_{1,j}, b \\ a, \delta R_{1,j}, b \end{pmatrix} + G\begin{pmatrix} a, \delta R_{1,j}, b \\ a, \delta R_{j+1,n}, b \end{pmatrix} \Bigg] \\
+ \frac{1}{(K^2)^2 s_{ab}^2} \Bigg[& G\begin{pmatrix} a, K_{1,j}, K_{j+1,n}, b \\ a, K_{1,j}, K_{j+1,n}, b \end{pmatrix} + 2G\begin{pmatrix} a, K_{1,j}, K_{1,n}, b \\ a, \delta R_{1,j}, K_{1,n}, b \end{pmatrix} + 2G\begin{pmatrix} a, K_{1,j}, K_{1,n}, b \\ a, \delta R_{j+1,n}, K_{1,n}, b \end{pmatrix} \\
& +G\begin{pmatrix} a, \delta R_{1,j}, K_{1,n}, b \\ a, \delta R_{1,j}, K_{1,n}, b \end{pmatrix} + 2G\begin{pmatrix} a, \delta R_{1,j}, K_{1,n}, b \\ a, \delta R_{j+1,n}, K_{1,n}, b \end{pmatrix} + G\begin{pmatrix} a, \delta R_{j+1,n}, K_{1,n}, b \\ a, \delta R_{j+1,n}, K_{1,n}, b \end{pmatrix} \Bigg] \\
\sim 1 + \mathcal{O}(\sqrt{\delta_a \delta_b})\,, &
\end{aligned}
\tag{8.6}
$$

so that

$$
\begin{aligned}
& k_{\hat{a}} + k_a + K_{1,j} \sim \\
& \quad - \frac{1}{2(K^2 - t_{1\cdots nb})} \Bigg[(\rho - 1)K^2 + 2t_{1\cdots nb} - 2K_{1,j}\cdot(2k_b + K_{1,n}) - 2\delta R_{1,j}\cdot(2k_b + K_{1,n}) \\
& \qquad - 2\delta R_{j+1,n}\cdot(2k_b + K_{1,n}) + \frac{1}{s_{ab}} G\begin{pmatrix} k_a, k_b \\ K_{1,j}, K_{j+1,n} \end{pmatrix} + \frac{1}{s_{ab}} G\begin{pmatrix} k_a, k_b \\ \delta R_{1,j}, K_{1,n} \end{pmatrix} \\
& \qquad + \frac{1}{s_{ab}} G\begin{pmatrix} k_a, k_b \\ \delta R_{j+1,n}, K_{1,n} \end{pmatrix} \Bigg] k_a + \delta R_{1,j} + \delta R_{j+1,n} \\
& \quad - \frac{1}{2(K^2 - t_{a1\cdots n})} \Bigg[(1-\rho)K^2 - 2K_{1,j}\cdot(2k_a + K_{1,n}) - 2\delta R_{1,j}\cdot(2k_a + K_{1,n}) \\
& \qquad - 2\delta R_{j+1,n}\cdot(2k_a + K_{1,n}) + \frac{1}{s_{ab}} G\begin{pmatrix} k_b, k_a \\ K_{1,j}, K_{1,n} \end{pmatrix} + \frac{1}{s_{ab}} G\begin{pmatrix} k_b, k_a \\ \delta R_{1,j}, K_{1,n} \end{pmatrix} \\
& \qquad + \frac{1}{s_{ab}} G\begin{pmatrix} k_b, k_a \\ \delta R_{j+1,n}, K_{1,n} \end{pmatrix} \Bigg] k_b
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2(K^2-t_{1\cdots nb})}\Bigg[(\rho-1)K^2 + 2(k_b+K_{j+1,n})^2 + 2K_{1,j}\cdot K_{j+1,n} - \frac{4k_a\cdot K_{j+1,n}\,k_b\cdot K_{1,j}}{s_{ab}} \\
&\qquad + \frac{4k_a\cdot K_{1,j}\,k_b\cdot K_{j+1,n}}{s_{ab}} - 2\delta R_{1,j}\cdot(2k_b+K_{1,n}) - 2\delta R_{j+1,n}\cdot(2k_b+K_{1,n}) \\
&\qquad + \frac{4k_a\cdot K_{1,j}\,k_b\cdot\delta R_{1,j}}{s_{ab}} + \frac{4k_a\cdot K_{j+1,n}\,k_b\cdot\delta R_{1,j}}{s_{ab}} - \frac{4k_a\cdot\delta R_{1,j}\,k_b\cdot K_{1,j}}{s_{ab}} \\
&\qquad - \frac{4k_a\cdot\delta R_{1,j}\,k_b\cdot K_{j+1,n}}{s_{ab}} + \frac{4k_a\cdot K_{1,j}\,k_b\cdot\delta R_{j+1,n}}{s_{ab}} + \frac{4k_a\cdot K_{j+1,n}\,k_b\cdot\delta R_{j+1,n}}{s_{ab}} \\
&\qquad - \frac{4k_a\cdot\delta R_{j+1,n}\,k_b\cdot K_{1,j}}{s_{ab}} - \frac{4k_a\cdot\delta R_{j+1,n}\,k_b\cdot K_{j+1,n}}{s_{ab}}\Bigg]k_a + \delta R_{1,j} + \delta R_{j+1,n} \\
&\quad - \frac{1}{2(K^2-t_{a1\cdots n})}\Bigg[(1-\rho)K^2 - 2K_{1,j}\cdot(2k_a+K_{1,j}) - 2K_{1,j}\cdot K_{j+1,n} - 2\delta R_{1,j}\cdot(2k_a+K_{1,n}) \\
&\qquad - 2\delta R_{j+1,n}\cdot(2k_a+K_{1,n}) - \frac{4k_b\cdot K_{j+1,n}\,k_a\cdot K_{1,j}}{s_{ab}} + \frac{4k_b\cdot K_{1,j}\,k_a\cdot K_{j+1,n}}{s_{ab}} \\
&\qquad - \frac{4k_b\cdot K_{1,j}\,k_a\cdot\delta R_{1,j}}{s_{ab}} - \frac{4k_b\cdot K_{j+1,n}\,k_a\cdot\delta R_{1,j}}{s_{ab}} + \frac{4k_b\cdot\delta R_{1,j}\,k_a\cdot K_{1,j}}{s_{ab}} \\
&\qquad + \frac{4k_b\cdot\delta R_{1,j}\,k_a\cdot K_{j+1,n}}{s_{ab}} - \frac{4k_b\cdot K_{1,j}\,k_a\cdot\delta R_{j+1,n}}{s_{ab}} - \frac{4k_b\cdot K_{j+1,n}\,k_a\cdot\delta R_{j+1,n}}{s_{ab}} \\
&\qquad + \frac{4k_b\cdot\delta R_{j+1,n}\,k_a\cdot K_{1,j}}{s_{ab}} + \frac{4k_b\cdot\delta R_{j+1,n}\,k_a\cdot K_{j+1,n}}{s_{ab}}\Bigg]k_b \\
&= \mathcal{O}(\sqrt{\delta_a\delta_b}) + \mathcal{O}(\delta_a) + \mathcal{O}(\delta_b)\,,
\end{aligned}
$$

with similar result for $k_b+k_{\hat b}+K_{j+1,m}$. A term that is of order δ_a or δ_b will kill off the leading pole in either $J(a,1,\ldots,j;\hat a;k_b+k_{\hat b}+K_{j+1,m})$ or $J(j+1,\ldots,m,b;\hat b;k_a+k_{\hat a}+K_{1,j})$, respectively. As in the double-emission case, the terms of order $\sqrt{\delta_a\delta_b}$ will either combine to kill off the leading poles in both currents, or else will kill off the leading, helicity-softened, square-root singularity in the 'canonical' part of the other current.

In the case $j=0$, we must consider the behavior of

$$k_a + k_{\hat a} \sim -\frac{(\rho-1)}{2}\,k_a + \frac{\rho-1}{2}\,k_b + \mathcal{O}(\delta_b)\,; \tag{8.7}$$

in this special case, $\rho-1 \sim \mathcal{O}(\delta_b^2)$, so that terms containing $k_a+k_{\hat a}$ will kill off the leading singularity in the current $J(1,\ldots,m,b;\hat b;k_a+k_{\hat a})$. A similar argument holds for the case $j=m$.

In all cases, the additional terms due to the excess momentum transferred between the two currents in each term will give no corrections to the leading singularity. We can thus set the excess momentum to zero in order to arrive at our final, simplified formula for the antenna amplitude for emission of m singular gluons,

$$\mathrm{Ant}(\hat a,\hat b \leftarrow a,1,\ldots,m,b) = \sum_{j=0}^{m} J(a,1,\ldots,j;\hat a;0)J(j+1,\ldots,m,b;\hat b;0)\,. \tag{8.8}$$

9. Antenna Amplitudes for Specific Helicities

Using eqn. (5.8) and the spinor-helicity basis [27], I obtain the following explicit forms for the antenna helicity amplitudes,

$$
\begin{aligned}
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, b^+) &= 0,\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, b^+) &= 0,\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, b^+) &= 0,\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, b^-) &= 0,\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^-, b^+) &= \frac{\langle a\, 1\rangle^3}{\langle a\, b\rangle\ \langle \hat{a}\, \hat{b}\rangle^2\ \langle 1\, b\rangle},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, b^-) &= \frac{\langle a\, b\rangle^3}{\langle a\, 1\rangle\ \langle \hat{a}\, \hat{b}\rangle^2\ \langle 1\, b\rangle},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, b^-) &= -\,\mathrm{Ant}(\hat{b}^+, \hat{a}^+ \leftarrow b^-, 1^-, a^+)\\
&= \frac{\langle 1\, b\rangle^3}{\langle a\, b\rangle\ \langle a\, 1\rangle\ \langle \hat{a}\, \hat{b}\rangle^2},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^-, b^-) &= -\frac{[\hat{a}\, \hat{b}]^2}{[a\, b]\ [a\, 1]\ [1\, b]},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^+, 1^+, b^+) &= 0,\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^-, 1^+, b^+) &= \frac{\langle a\, \hat{b}\rangle^4}{\langle a\, b\rangle\ \langle a\, 1\rangle\ \langle \hat{a}\, \hat{b}\rangle^2\ \langle 1\, b\rangle},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^+, 1^-, b^+) &= -\frac{\langle 1\, \hat{b}\rangle^3\ [\hat{a}\, b]}{\langle a\, b\rangle\ \langle a\, 1\rangle\ \langle \hat{a}\, \hat{b}\rangle^2\ [\hat{a}\, \hat{b}]},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^+, 1^+, b^-) &= 0,\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^-, 1^-, b^+) &= -\frac{[\hat{a}\, b]^4}{[a\, b]\ [a\, 1]\ [\hat{a}\, \hat{b}]^2\ [1\, b]},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^-, 1^+, b^-) &= -\frac{[a\, 1]^3}{[a\, b]\ [\hat{a}\, \hat{b}]^2\ [1\, b]},\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^+, 1^-, b^-) &= 0,\\
\mathrm{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a^-, 1^-, b^-) &= 0.
\end{aligned}
\tag{9.1}
$$

The remaining helicity amplitudes can be obtained via parity or reflection antisymmetry. In deriving these forms, I have used identities such as

$$
\frac{\langle q\, \hat{a}\rangle}{\langle q\, a\rangle} = \frac{\langle b\, \hat{a}\rangle}{\langle b\, a\rangle} + \frac{\langle q\, b\rangle\, \langle \hat{a}\, a\rangle}{\langle q\, a\rangle\, \langle b\, a\rangle}, \tag{9.2}
$$

and have dropped non-universal terms (terms insufficiently singular in the various limits).

From eqn. (7.17), I obtain the $2 \leftarrow 4$ antenna helicity amplitudes,

$$
\begin{aligned}
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^+,2^+,b^+) = 0\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^+,2^+,b^+) = 0\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^-,2^+,b^+) = 0\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^+,2^-,b^+) = 0\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^+,2^+,b^-) = 0\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^-,2^+,b^+) = -\frac{\langle a\,1\rangle^2\,\langle \hat{a}\,1\rangle\,[\hat{a}\,b]}{\langle a\,b\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,b]}\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^+,2^-,b^+) = \frac{\langle a\,2\rangle^4}{\langle a\,b\rangle\,\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,\langle 2\,b\rangle}\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^+,2^+,b^-) = \frac{\langle a\,b\rangle^3}{\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,\langle 2\,b\rangle}\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^-,2^-,b^+) = \frac{\langle \hat{a}\,2\rangle^3\,\langle 1\,\hat{b}\rangle^3}{\langle a\,b\rangle\,\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle^5\,\langle 2\,b\rangle}\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^-,2^+,b^-) = \mathrm{Ant}(\hat{b}^+,\hat{a}^+ \leftarrow b^-,2^+,1^-,a^+)\\
&\qquad = \frac{\langle 1\,b\rangle^4}{\langle a\,b\rangle\,\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,\langle 2\,b\rangle}\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^+,2^-,b^-) = \mathrm{Ant}(\hat{b}^+,\hat{a}^+ \leftarrow b^-,2^-,1^+,a^+)\\
&\qquad = \frac{\langle 2\,b\rangle^3}{\langle a\,b\rangle\,\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle}\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^-,2^-,b^+) =\\
&\quad \frac{\langle a\,1\rangle^2\,\langle a\,2\rangle^2\,[a\,b]}{(s_{ab}+s_{a2})\,s_{2b}\,\langle a\,b\rangle\,\langle \hat{a}\,\hat{b}\rangle^2} + \frac{\langle a\,2\rangle\,\langle \hat{a}\,2\rangle\,[\hat{a}\,b]\,(s_{\hat{a}1}+s_{\hat{a}2})}{\langle a\,b\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 2\,b\rangle\,[a\,b]\,[a\,1]\,[1\,2]}\\
&\quad + \frac{\langle a\,\hat{b}\rangle\,\langle a\,2\rangle\,\langle \hat{a}\,1\rangle\,[\hat{a}\,b]\,[b\,\hat{b}]}{\langle a\,b\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 2\,b\rangle\,[a\,b]\,[1\,2]\,[2\,b]} + \frac{\langle a\,\hat{b}\rangle\,\langle 1\,2\rangle\,[\hat{a}\,b]\,[b\,\hat{b}]^2}{\langle \hat{a}\,\hat{b}\rangle\,\langle 2\,b\rangle\,[a\,b]\,[1\,2]\,[2\,b]\,t_{12b}}\,,\\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^-,2^+,b^-) =\\
&\quad -\frac{\langle a\,b\rangle^2\,\langle 1\,b\rangle\,[a\,2]\,[\hat{a}\,\hat{b}]^2}{s_{ab}\,s_{\hat{a}\hat{b}}{}^2\,\langle 2\,b\rangle\,[a\,1]} - \frac{\langle a\,b\rangle^2\,\langle 1\,b\rangle\,[a\,\hat{b}]\,[\hat{a}\,\hat{b}]\,[\hat{a}\,2]}{s_{ab}\,s_{\hat{a}\hat{b}}\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,1]\,[1\,2]} - \frac{\langle a\,b\rangle^2\,\langle \hat{a}\,1\rangle\,[a\,\hat{b}]\,[\hat{a}\,2]^2}{\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 2\,b\rangle\,[a\,b]\,[a\,1]\,[\hat{a}\,\hat{b}]\,[1\,2]}\\
&\quad + \frac{\langle a\,1\rangle^2\,\langle \hat{a}\,b\rangle\,[a\,\hat{b}]\,[\hat{a}\,2]^2}{(s_{ab}+s_{a2})\,\langle \hat{a}\,\hat{b}\rangle^2\,\langle 2\,b\rangle\,[a\,b]\,[\hat{a}\,\hat{b}]\,[2\,b]} + \frac{(s_{1b}-s_{a1})\,\langle a\,b\rangle\,\langle 1\,b\rangle\,[\hat{a}\,\hat{b}]\,[\hat{a}\,2]\,[2\,\hat{b}]}{s_{ab}\,s_{\hat{a}\hat{b}}\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,1]\,[1\,2]\,[2\,b]}\\
&\quad + \frac{\langle a\,1\rangle\,\langle \hat{a}\,b\rangle\,[a\,\hat{b}]\,[\hat{a}\,2]^2}{\langle \hat{a}\,\hat{b}\rangle\,\langle 1\,2\rangle\,[a\,b]\,[a\,1]\,[1\,2]\,t_{a12}} + \frac{\langle a\,\hat{b}\rangle\,\langle 1\,b\rangle^2\,[\hat{a}\,b]\,[2\,\hat{b}]^2}{\langle \hat{a}\,\hat{b}\rangle\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,b]\,[1\,2]\,[2\,b]\,t_{12b}}\,,
\end{aligned}
\tag{9.3}
$$

$$
\begin{aligned}
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^+,2^-,b^-) = \mathrm{Ant}(\hat{b}^+,\hat{a}^+ \leftarrow b^-,2^-,1^+,a^-) = \\
&\quad \frac{\langle a\,\hat{b}\rangle\;\langle 2\,b\rangle^2\;[\hat{a}\,b]\;[1\,\hat{b}]^2}{(s_{ab}+s_{1b})\;\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\;[a\,b]\;[a\,1]\;[\hat{a}\,\hat{b}]} - \frac{\langle a\,b\rangle^2\;\langle a\,2\rangle\;[\hat{a}\,\hat{b}]^2\;[1\,b]}{s_{ab}\,s_{\hat{a}\hat{b}}{}^2\;\langle a\,1\rangle\;[2\,b]} - \frac{\langle a\,b\rangle^2\;\langle a\,2\rangle\;[\hat{a}\,b]\;[\hat{a}\,\hat{b}]\;[1\,\hat{b}]}{s_{ab}\,s_{\hat{a}\hat{b}}\;\langle a\,1\rangle\;\langle 1\,2\rangle\;[1\,2]\;[2\,b]} \\
&\quad + \frac{(s_{a2}-s_{2b})\;\langle a\,b\rangle\;\langle a\,2\rangle\;[\hat{a}\,\hat{b}]\;[\hat{a}\,1]\;[1\,\hat{b}]}{s_{ab}\,s_{\hat{a}\hat{b}}\;\langle a\,1\rangle\;\langle 1\,2\rangle\;[a\,1]\;[1\,2]\;[2\,b]} - \frac{\langle a\,b\rangle^2\;\langle 2\,\hat{b}\rangle\;[\hat{a}\,b]\;[1\,\hat{b}]^2}{\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\;\langle 2\,b\rangle\;[a\,b]\;[\hat{a}\,\hat{b}]\;[1\,2]\;[2\,b]} \\
&\quad + \frac{\langle a\,2\rangle^2\;\langle \hat{a}\,b\rangle\;[a\,\hat{b}]\;[\hat{a}\,1]^2}{\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle\;\langle 1\,2\rangle\;[a\,b]\;[a\,1]\;[1\,2]\;t_{a12}} + \frac{\langle a\,\hat{b}\rangle\;\langle 2\,b\rangle\;[\hat{a}\,b]\;[1\,\hat{b}]^2}{\langle \hat{a}\,\hat{b}\rangle\;\langle 1\,2\rangle\;[a\,b]\;[1\,2]\;[2\,b]\;t_{12b}}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^+,1^-,2^-,b^-) = \mathrm{Ant}(\hat{b}^+,\hat{a}^+ \leftarrow b^-,2^-,1^-,a^+) = \\
&\quad \frac{\langle 1\,b\rangle^2\;\langle 2\,b\rangle^2\;[a\,b]}{s_{a1}\;(s_{ab}+s_{1b})\;\langle a\,b\rangle\;\langle \hat{a}\,\hat{b}\rangle^2} - \frac{\langle \hat{a}\,b\rangle\;\langle 1\,b\rangle\;\langle 2\,\hat{b}\rangle\;[a\,\hat{a}]\;[a\,\hat{b}]}{\langle a\,b\rangle\;\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\;[a\,b]\;[a\,1]\;[1\,2]} \\
&\quad - \frac{\langle 1\,b\rangle\;\langle 1\,\hat{b}\rangle\;[a\,\hat{b}]\;(\langle 1\,\hat{b}\rangle\;[1\,\hat{b}]+\langle 2\,\hat{b}\rangle\;[2\,\hat{b}])}{\langle a\,b\rangle\;\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\;[a\,b]\;[1\,2]\;[2\,b]} + \frac{\langle \hat{a}\,b\rangle\;\langle 1\,2\rangle\;[a\,\hat{a}]^2\;[a\,\hat{b}]}{\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle\;[a\,b]\;[a\,1]\;[1\,2]\;t_{a12}}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^+ \leftarrow a^-,1^-,2^-,b^-) = \frac{[\hat{a}\,\hat{b}]^2}{[a\,b]\;[a\,1]\;[1\,2]\;[2\,b]}\,,
\end{aligned}
\tag{9.4}
$$

and

$$
\begin{aligned}
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^+,2^+,b^+) = 0\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^+,2^+,b^+) = \frac{\langle a\,\hat{b}\rangle^4}{\langle a\,b\rangle\,\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,\langle 2\,b\rangle}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^-,2^+,b^+) = \frac{\langle 1\,\hat{b}\rangle^4}{\langle a\,b\rangle\,\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,\langle 2\,b\rangle}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^+,2^-,b^+) = -\frac{\langle 2\,\hat{b}\rangle^3\,[\hat{a}\,b]}{\langle a\,b\rangle\,\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,[\hat{a}\,\hat{b}]}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^+,2^+,b^-) = 0\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^-,2^+,b^+) = \\
&\quad \frac{(s_{\hat{a}\hat{b}}-s_{a1}-s_{2b})\,\langle a\,\hat{b}\rangle\,\langle 1\,\hat{b}\rangle\,[\hat{a}\,b]\,[\hat{a}\,2]}{s_{ab}s_{\hat{a}\hat{b}}\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,1]\,[1\,2]} + \frac{\langle a\,\hat{b}\rangle\,\langle a\,1\rangle\,[\hat{a}\,b]\,[\hat{a}\,2]^2}{\langle a\,b\rangle\,\langle 1\,2\rangle\,[a\,1]\,[\hat{a}\,\hat{b}]\,[1\,2]\;t_{a12}} + \frac{\langle a\,\hat{b}\rangle\;\langle 1\,\hat{b}\rangle^2\,[\hat{a}\,b]\,[2\,b]}{\langle \hat{a}\,\hat{b}\rangle\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,b]\,[1\,2]\;t_{12b}}\,,
\end{aligned}
\tag{9.5}
$$

$$
\begin{aligned}
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^+,2^-,b^+) = \\
&\quad \frac{\langle a\,\hat{b}\rangle\;\langle 2\,\hat{b}\rangle^2\,[\hat{a}\,b]\;[1\,b]^2}{(s_{ab}+s_{1b})\,\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,[a\,b]\,[a\,1]\,[\hat{a}\,\hat{b}]} - \frac{\langle a\,\hat{b}\rangle\;\langle a\,2\rangle^2\,[\hat{a}\,b]^3}{\langle a\,b\rangle\,\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle\,\langle 1\,2\rangle\,[a\,b]\;[\hat{a}\,\hat{b}]^2\,[2\,b]} \\
&\quad + \frac{\langle a\,\hat{b}\rangle\;\langle a\,2\rangle^2\,[\hat{a}\,b]\;[\hat{a}\,1]^2}{(s_{ab}+s_{a2})\,\langle a\,b\rangle\,\langle \hat{a}\,\hat{b}\rangle\,\langle 2\,b\rangle\;[\hat{a}\,\hat{b}]^2\,[2\,b]} - \frac{\langle a\,\hat{b}\rangle^2\,\langle a\,2\rangle\;[\hat{a}\,b]^2\,[1\,b]}{s_{ab}\,s_{\hat{a}\hat{b}}\,\langle a\,1\rangle\,\langle 1\,2\rangle\,[1\,2]\,[2\,b]} \\
&\quad + \frac{(s_{12}-s_{a1}-s_{2b})\,\langle a\,\hat{b}\rangle\,\langle a\,2\rangle\,\langle 2\,\hat{b}\rangle\,[\hat{a}\,b]\,[\hat{a}\,1]\,[1\,b]}{s_{ab}\,s_{\hat{a}\hat{b}}\,\langle a\,1\rangle\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,1]\,[1\,2]\,[2\,b]} - \frac{\langle a\,\hat{b}\rangle^3\,[\hat{a}\,b]\;[1\,b]^2}{\langle a\,b\rangle\,\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,[a\,b]\,[\hat{a}\,\hat{b}]\,[1\,2]\,[2\,b]} \\
&\quad + \frac{\langle a\,\hat{b}\rangle\;\langle a\,2\rangle^2\,[\hat{a}\,b]\;[\hat{a}\,1]^2}{\langle a\,b\rangle\,\langle a\,1\rangle\,\langle 1\,2\rangle\,[a\,1]\,[\hat{a}\,\hat{b}]\,[1\,2]\;t_{a12}} + \frac{\langle a\,\hat{b}\rangle\;\langle 2\,\hat{b}\rangle^2\,[\hat{a}\,b]\;[1\,b]^2}{\langle \hat{a}\,\hat{b}\rangle\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,b]\,[1\,2]\,[2\,b]\;t_{12b}}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^+,2^+,b^-) = \\
&\quad \frac{\langle a\,b\rangle\;\langle a\,\hat{b}\rangle^2\,[a\,2]^2\,[\hat{a}\,1]^2}{(s_{ab}+s_{a2})\;s_{2b}\,\langle \hat{a}\,\hat{b}\rangle^2\,[a\,b]\;[\hat{a}\,\hat{b}]^2} - \frac{\langle a\,\hat{b}\rangle^2\,\langle b\,\hat{b}\rangle\,[a\,2]\,[\hat{a}\,1]}{\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,b]\,[\hat{a}\,\hat{b}]\,[2\,b]} - \frac{\langle a\,\hat{b}\rangle^2\,[a\,2]\,\langle a^-|\,b+\hat{b}\,|\hat{a}^-\rangle\,[\hat{a}\,2]}{\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,\langle 1\,2\rangle\,[a\,b]\;[\hat{a}\,\hat{b}]^2\,[2\,b]} \\
&\quad + \frac{\langle a\,\hat{b}\rangle\;\langle b\,\hat{b}\rangle^2\,[\hat{a}\,b]\,[1\,2]}{\langle \hat{a}\,\hat{b}\rangle\,\langle 1\,2\rangle\,\langle 2\,b\rangle\,[a\,b]\,[2\,b]\;t_{12b}}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^-,2^-,b^+) = \\
&\quad \frac{\langle 1\,b\rangle^2\,\langle 2\,\hat{b}\rangle^2\,[a\,b]\;[\hat{a}\,b]^2}{s_{a1}\,(s_{ab}+s_{1b})\,\langle a\,b\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,[\hat{a}\,\hat{b}]^2} + \frac{\langle 1\,b\rangle\,\langle 2\,\hat{b}\rangle\,[a\,\hat{a}]\;[\hat{a}\,b]^2}{\langle a\,b\rangle\,\langle a\,1\rangle\,\langle \hat{a}\,\hat{b}\rangle\,[a\,1]\;[\hat{a}\,\hat{b}]^2\,[1\,2]} - \frac{\langle 1\,b\rangle\,\langle 1\,\hat{b}\rangle\;[\hat{a}\,b]^2\,\langle \hat{b}^-|\,a+\hat{a}\,|b^-\rangle}{\langle a\,b\rangle\,\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,[\hat{a}\,\hat{b}]^2\,[1\,2]\,[2\,b]} \\
&\quad + \frac{\langle a\,\hat{b}\rangle\,\langle 1\,2\rangle\;[a\,\hat{a}]^2\,[\hat{a}\,b]}{\langle a\,b\rangle\,\langle a\,1\rangle\,[a\,1]\,[\hat{a}\,\hat{b}]\,[1\,2]\;t_{a12}}\,,
\end{aligned}
\tag{9.6}
$$

$$
\begin{aligned}
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^-,2^+,b^-) = -\frac{\langle 1\,b\rangle\;\langle 1\,\hat{b}\rangle^2\,[a\,2]\;[\hat{a}\,2]^2}{\langle a\,b\rangle\,\langle a\,1\rangle\;\langle \hat{a}\,\hat{b}\rangle^2\,[a\,b]\;[\hat{a}\,\hat{b}]^2\,[2\,b]}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^+,2^-,b^-) = 0\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^-,2^-,b^+) = \frac{[\hat{a}\,b]^4}{[a\,b]\,[a\,1]\;[\hat{a}\,\hat{b}]^2\,[1\,2]\,[2\,b]}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^-,2^+,b^-) = \frac{[\hat{a}\,2]^4}{[a\,b]\,[a\,1]\;[\hat{a}\,\hat{b}]^2\,[1\,2]\,[2\,b]}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^+,2^-,b^-) = -\frac{\langle a\,\hat{b}\rangle\;[\hat{a}\,1]^3}{\langle \hat{a}\,\hat{b}\rangle\,[a\,b]\;[\hat{a}\,\hat{b}]^2\,[1\,2]\,[2\,b]}\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^+,1^-,2^-,b^-) = 0\,, \\
&\mathrm{Ant}(\hat{a}^+,\hat{b}^- \leftarrow a^-,1^-,2^-,b^-) = 0\,,
\end{aligned}
\tag{9.7}
$$

The reader may verify that these functions reproduce the appropriate triple-collinear, products of collinear splitting amplitudes, mixed soft-collinear, or double-soft amplitudes in the various limits listed in section 7.

10. The Antenna Amplitudes Squared in Dimensional Regularization

If we square the expressions for the single-emission antenna functions given in the previous section, and sum over the helicities of legs $a, 1, b$ while averaging over the helicities of $\hat{a}$ and $\hat{b}$, we obtain the following expression in four dimensions,

$$|\,\mathrm{Ant}_1\,|^2 = 2\frac{\left(K^2(s_{a1}+s_{1b})+s_{ab}^2\right)^2}{s_{a1}s_{1b}s_{ab}(K^2)^2} \tag{10.1}$$

The same expression turns out to hold away from $D=4$, in the conventional dimensional regularization (CDR) scheme [28]. (In the CDR scheme, there are $D-2$ gluon helicities in contrast to the FDH scheme [29,30] in which the number of gluon helicities is kept fixed at 2.)

For the double-emission antenna function, I find for the helicity-summed and -averaged square in four dimensions,

$$|\,\mathrm{Ant}_2\,|^2 = \frac{1}{4}\left[A_1(\hat{a},a,1,2,b,\hat{b}) + A_2(\hat{a},a,1,2,b,\hat{b}) + A_2(\hat{b},b,2,1,a,\hat{a})\right]\,, \tag{10.2}$$

where

$$\begin{aligned}
&A_1(\hat{a},a,1,2,b,\hat{b}) = \\
&\quad \frac{16\,s_{a1}\,s_{2b}}{s_{12}^2\,t_{a12}\,t_{12b}} - \frac{16 s_{\hat{a}\hat{b}}}{s_{12}\,t_{a12}\,t_{12b}}\left(2 - \frac{(s_{a1}+s_{2b})}{s_{\hat{a}\hat{b}}} + \frac{2\,(s_{a1}+s_{2b})^2}{{s_{\hat{a}\hat{b}}}^2}\right) - \frac{32}{t_{a12}\,t_{12b}} \\
&\quad + \frac{1}{s_{a1}\,s_{2b}}\left(42 + \frac{8\,s_{ab}^2}{{s_{\hat{a}\hat{b}}}^2} - \frac{36\,s_{ab}}{s_{\hat{a}\hat{b}}} - \frac{70\,s_{\hat{a}\hat{b}}}{s_{ab}} + \frac{46\,t_{a12}}{s_{ab}} + \frac{32\,s_{ab}\,t_{a12}}{{s_{\hat{a}\hat{b}}}^2} - \frac{16\,t_{a12}}{s_{\hat{a}\hat{b}}} + \frac{16\,t_{a12}^2}{{s_{\hat{a}\hat{b}}}^2} - \frac{32\,t_{a12}^2}{s_{ab}\,s_{\hat{a}\hat{b}}}\right. \\
&\qquad\qquad \left. + \frac{16\,t_{a12}^3}{s_{ab}\,{s_{\hat{a}\hat{b}}}^2} + \frac{46\,t_{12b}}{s_{ab}} + \frac{32\,s_{ab}\,t_{12b}}{{s_{\hat{a}\hat{b}}}^2} - \frac{16\,t_{12b}}{s_{\hat{a}\hat{b}}} + \frac{16\,t_{12b}^2}{{s_{\hat{a}\hat{b}}}^2} - \frac{32\,t_{12b}^2}{s_{ab}\,s_{\hat{a}\hat{b}}} + \frac{16\,t_{12b}^3}{s_{ab}\,{s_{\hat{a}\hat{b}}}^2}\right) \\
&\quad + \frac{s_{\hat{a}\hat{b}}}{s_{a1}\,s_{12}\,s_{2b}}\left(\frac{28\,s_{ab}}{s_{\hat{a}\hat{b}}} + \frac{8\,s_{ab}^3}{{s_{\hat{a}\hat{b}}}^3} - \frac{20\,s_{ab}^2}{{s_{\hat{a}\hat{b}}}^2} - 16 + \frac{8\,s_{\hat{a}\hat{b}}}{s_{ab}}\right) + \frac{2\,{s_{\hat{a}\hat{b}}}^2}{s_{a1}\,s_{2b}\,t_{a12}\,t_{12b}}\left(4 + \frac{2\,s_{12}}{s_{\hat{a}\hat{b}}} + \frac{s_{12}^2}{s_{ab}s_{\hat{a}\hat{b}}}\right) \\
&\quad + \frac{1}{s_{12}^2}\left(-34 - \frac{10\,s_{ab}^2}{{s_{\hat{a}\hat{b}}}^2} - \frac{14\,s_{ab}\,s_{a1}}{{s_{\hat{a}\hat{b}}}^2} - \frac{8\,{s_{a1}}^2}{{s_{\hat{a}\hat{b}}}^2} + \frac{36\,s_{ab}}{s_{\hat{a}\hat{b}}} + \frac{38\,s_{a1}}{s_{\hat{a}\hat{b}}} - \frac{14\,s_{ab}\,s_{2b}}{{s_{\hat{a}\hat{b}}}^2} + \frac{38\,s_{2b}}{s_{\hat{a}\hat{b}}} - \frac{8\,s_{2b}^2}{{s_{\hat{a}\hat{b}}}^2}\right),
\end{aligned} \tag{10.3}$$

and

$$
\begin{aligned}
&A_2(\hat{a},a,1,2,b,\hat{b}) = \\
&\frac{4\,{s_{\hat{a}\hat{b}}}^4}{s_{a1}^2\,s_{12}^2\,t_{a12}^2}\left(\frac{s_{a2}^2\,{s_{\hat{a}1}}^2}{{s_{\hat{a}\hat{b}}}^4}+\frac{s_{a1}^2\,{s_{\hat{a}2}}^2}{{s_{\hat{a}\hat{b}}}^4}+\frac{s_{a\hat{a}}^2\,s_{12}^2}{{s_{\hat{a}\hat{b}}}^4}\right)-\frac{8\,s_{\hat{a}\hat{b}}}{s_{a1}\,t_{a12}\,t_{12b}}\left(\frac{\left(s_{\hat{a}\hat{b}}-s_{2b}\right)^3}{{s_{\hat{a}\hat{b}}}^3}+\frac{{s_{\hat{a}\hat{b}}}^2-s_{2b}^2}{{s_{\hat{a}\hat{b}}}^2}\right)\\
&+\frac{4}{s_{a1}^2}\left(-9-\frac{2\,s_{a\hat{b}}^2}{{s_{\hat{a}\hat{b}}}^2}-\frac{10\,s_{a\hat{b}}}{s_{\hat{a}\hat{b}}}-\frac{4\,s_{1b}\,s_{1\hat{b}}}{s_{\hat{a}\hat{b}}\,\left(s_{ab}+s_{1b}\right)}-\frac{4\,s_{ab}\,s_{2b}}{{s_{\hat{a}\hat{b}}}^2}+\frac{10\,s_{2b}}{s_{\hat{a}\hat{b}}}-\frac{3\,s_{2b}^2}{{s_{\hat{a}\hat{b}}}^2}+\frac{2\,s_{1b}^2\,s_{2b}^2}{{s_{\hat{a}\hat{b}}}^2\left(s_{ab}+s_{1b}\right)^2}\right.\\
&\qquad\left.+\frac{2\,s_{a\hat{b}}\,s_{1\hat{b}}}{{s_{\hat{a}\hat{b}}}^2}+\frac{2\,s_{a\hat{b}}\,s_{1\hat{b}}\,t_{\hat{a}a1}}{s_{2b}\,t_{a12}\,s_{a\hat{b}}}\right)-\frac{8\,\left(s_{\hat{a}\hat{b}}-s_{1\hat{b}}\right)\,s_{1\hat{b}}\,t_{a12}}{s_{a1}^2\,{s_{\hat{a}\hat{b}}}^2\,s_{12}}+\frac{s_{a1}\,\left(11\,t_{12b}-16\,s_{2b}\right)}{s_{\hat{a}\hat{b}}\,s_{12}^2\,t_{a12}}-\frac{4\,s_{ab}\,s_{\hat{a}1}}{s_{a1}\,s_{12}\,s_{2b}\,t_{a12}}\\
&+\frac{2}{s_{a1}\,s_{12}}\left(-3+\frac{s_{ab}^2}{{s_{\hat{a}\hat{b}}}^2}+\frac{8\,s_{ab}}{s_{\hat{a}\hat{b}}}-\frac{8\,s_{\hat{a}\hat{b}}}{s_{ab}}+\frac{12\,s_{2b}}{s_{ab}}+\frac{9\,s_{ab}\,s_{2b}}{{s_{\hat{a}\hat{b}}}^2}-\frac{5\,s_{2b}}{s_{\hat{a}\hat{b}}}+\frac{12\,s_{2b}^2}{{s_{\hat{a}\hat{b}}}^2}-\frac{8\,s_{2b}^2}{s_{ab}\,s_{\hat{a}\hat{b}}}+\frac{4\,s_{2b}^3}{s_{ab}\,{s_{\hat{a}\hat{b}}}^2}\right.\\
&\qquad\left.+\frac{4\,\left({s_{\hat{a}\hat{b}}}^2+s_{1b}^2\right)\,t_{12b}}{{s_{\hat{a}\hat{b}}}^2\,\left(s_{ab}+s_{1b}\right)}\right)-\frac{4\,s_{1\hat{b}}^2\,t_{a12}^2}{s_{a1}^2\,{s_{\hat{a}\hat{b}}}^2\,s_{12}^2}+\frac{4\,s_{\hat{a}\hat{b}}}{s_{a1}\,s_{2b}\,t_{a12}}\left(\frac{s_{ab}}{s_{\hat{a}\hat{b}}}-\frac{2\,s_{ab}^2}{{s_{\hat{a}\hat{b}}}^2}-4+\frac{2\,s_{\hat{a}\hat{b}}}{s_{ab}}\right)\\
&+\frac{s_{\hat{a}\hat{b}}}{s_{a1}\,s_{12}^2}\left(\frac{-8\,s_{2b}\,t_{a12}}{{s_{\hat{a}\hat{b}}}^2}-\frac{8\,s_{1b}\,s_{2b}\,t_{a12}}{{s_{\hat{a}\hat{b}}}^3}-\frac{8\,s_{ab}\,t_{a12}\,t_{12b}}{{s_{\hat{a}\hat{b}}}^3}+\frac{16\,t_{a12}\,t_{12b}}{{s_{\hat{a}\hat{b}}}^2}-\frac{8\,s_{2b}\,t_{a12}\,t_{12b}}{{s_{\hat{a}\hat{b}}}^3}+\frac{3\,s_{2b}^2\,t_{a12}\,t_{12b}}{{s_{\hat{a}\hat{b}}}^4}\right.\\
&\qquad\left.-\frac{6\,s_{2b}\,t_{a12}\,t_{12b}^2}{{s_{\hat{a}\hat{b}}}^4}+\frac{3\,t_{a12}\,t_{12b}^3}{{s_{\hat{a}\hat{b}}}^4}\right)-\frac{8\,s_{ab}\,t_{\hat{a}a1}\,s_{1b}}{s_{a1}^2\,s_{\hat{a}\hat{b}}\,s_{2b}\,t_{a12}}-\frac{3\,{s_{\hat{a}\hat{b}}}^2}{s_{a1}\,s_{12}^2\,t_{a12}}\left(\frac{s_{\hat{a}1}^2}{{s_{\hat{a}\hat{b}}}^2}+\frac{s_{a\hat{b}}\,s_{\hat{a}1}^2}{{s_{\hat{a}\hat{b}}}^3}\right)\\
&+\frac{2}{s_{12}\,t_{a12}}\left(16+\frac{8\,s_{2b}}{s_{ab}}+\frac{4\,s_{2b}}{s_{\hat{a}\hat{b}}}-\frac{4\,s_{2b}^2}{s_{ab}\,s_{\hat{a}\hat{b}}}+\frac{4\,s_{2b}^2}{s_{ab}\,\left(s_{ab}+s_{1b}\right)}-\frac{8\,\left(s_{1b}+s_{2b}\right)}{s_{ab}+s_{1b}}-\frac{4\,s_{1b}\,\left(s_{1b}+s_{2b}\right)^2}{{s_{\hat{a}\hat{b}}}^2\,\left(s_{ab}+s_{1b}\right)}\right.\\
&\qquad\left.-\frac{4\,t_{12b}}{s_{ab}}+\frac{3\,t_{12b}}{s_{\hat{a}\hat{b}}}+\frac{4\,t_{12b}^2}{{s_{\hat{a}\hat{b}}}^2}\right)+\frac{8\,s_{\hat{a}\hat{b}}}{s_{a1}^2\,t_{a12}}\left(\frac{-2\,s_{12}\,s_{1b}}{\left(s_{ab}+s_{1b}\right)s_{\hat{a}\hat{b}}}+\frac{s_{12}\,\left(s_{\hat{a}\hat{b}}-2s_{1\hat{b}}+s_{2b}\right)}{{s_{\hat{a}\hat{b}}}^2}\right)\\
&+\frac{4\,s_{\hat{a}\hat{b}}}{s_{12}\,s_{2b}\,t_{a12}}\left(\frac{-2\,s_{a1}^3}{{s_{\hat{a}\hat{b}}}^3}+\frac{4\,s_{a1}^2}{{s_{\hat{a}\hat{b}}}^2}+\frac{2\,s_{a1}\,{s_{a2}}^2}{\left(s_{ab}+s_{a2}\right)\,{s_{\hat{a}\hat{b}}}^2}+\frac{s_{ab}}{s_{\hat{a}\hat{b}}}-\frac{2\,s_{a1}}{s_{\hat{a}\hat{b}}}+\frac{2\,s_{ab}\,t_{12b}^2}{{s_{\hat{a}\hat{b}}}^3}+\frac{2\,t_{12b}^2}{s_{ab}\,s_{\hat{a}\hat{b}}}\right)\\
&-\frac{1}{s_{a1}\,t_{a12}}\left(\frac{16\,s_{\hat{a}\hat{b}}}{s_{ab}}+\frac{16s_{1b}\,\left(s_{\hat{a}\hat{b}}+2s_{1b}\right)}{s_{\hat{a}\hat{b}}\,\left(s_{ab}+s_{1b}\right)}-\frac{24\,s_{2b}}{s_{ab}}+\frac{32\,s_{2b}}{s_{\hat{a}\hat{b}}}+\frac{32\,s_{2b}}{s_{ab}+s_{1b}}+\frac{16\,s_{2b}^2}{s_{ab}\,s_{\hat{a}\hat{b}}}-\frac{8\,s_{2b}^3}{s_{ab}\,{s_{\hat{a}\hat{b}}}^2}-\frac{59\,t_{12b}}{s_{\hat{a}\hat{b}}}\right).
\end{aligned}
\tag{10.4}
$$

Outside of four dimensions, in the CDR scheme there are the following additional contributions,

$$
\delta|\,\mathrm{Ant}_2\,|^2=-2\epsilon\left[E_1(\hat{a},a,1,2,b,\hat{b})+E_2(\hat{a},a,1,2,b,\hat{b})+E_2(\hat{b},b,2,1,a,\hat{a})\right]\,, \tag{10.5}
$$

where

$$
\begin{aligned}
&E_1(\hat{a},a,1,2,b,\hat{b})=\frac{2\,\left(s_{a1}+s_{12}\right)\,\left(s_{12}+s_{2b}\right)}{s_{12}^2\,t_{a12}\,t_{12b}}+\frac{1}{s_{12}^2}\,,\\
&E_2(\hat{a},a,1,2,b,\hat{b})=\\
&\quad\frac{\left(s_{a1}^2+s_{a1}\,s_{12}+s_{12}^2\right)^2}{s_{a1}^2\,s_{12}^2\,t_{a12}^2}-\frac{2\,\left(s_{a1}+s_{12}\right)\,\left(s_{a1}^2+s_{12}^2\right)}{s_{a1}^2\,s_{12}^2\,t_{a12}}+\frac{2\,s_{ab}\,\left(s_{a1}+s_{12}\right)}{s_{a1}^2\,\left(s_{ab}+s_{1b}\right)\,t_{a12}}+\frac{s_{1b}^2}{s_{a1}^2(s_{ab}+s_{1b})^2}\,.
\end{aligned}
\tag{10.6}
$$

11. Summary

This paper provides a general formula (8.8) for a function, an *antenna amplitude*, summarizing all multiply-collinear, mixed collinear-soft, and multiple-soft singularities in the emission of color-connected gluons in an arbitrary tree-level amplitude in gauge theory. Products of such functions over separate color-connected sets will then describe all singular limits of tree-level amplitudes. I have also evaluated all the antenna helicity amplitudes for single- and double-gluon emission, and given expressions for the helicity-summed and -averaged forms, both in and away from four dimensions. The latter functions, integrated appropriately over the phase space for singular emission, would provide functions that will cancel off virtual singularities in loop amplitudes. Integrals over singular phase space of the single-emission functions (10.1) would combine the soft and collinear integrals of Giele and Glover [31], and be equivalent to the integrated dipole functions of Catani and Seymour [14]. Integrals of the double-emission functions (10.2,10.5) would provide the corresponding ingredient in NNLO calculations. The final-state integration would be sufficient for the simplest NNLO process, $e^+e^- \to 3$ jets. Although this paper presented results only for pure-gluon processes, the master formulæ (5.8,7.17,8.8) carry over to mixed quark-gluon antenna amplitudes simply with the replacement of gluon currents by the quark equivalents as appropriate, and the reconstruction functions (5.3,5.7,6.2,7.8) carry over unchanged. Supersymmetry identities [32,26] can of course also be used to relate the quark and gluon antenna amplitudes.

The construction of the antenna amplitude relies on the reconstruction functions, which combine sets of momenta into a pair of massless momenta, with smooth limits as additional gluons' momenta become soft or collinear. These reconstruction functions (or more precisely their inverses) can be used to generate near-singular phase-space configurations numerically in an efficient manner, and thus should prove useful in the writing of numerical programs for higher-order calculations as well. Ref. [33] gives an example of a similar remapping.

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Appendix I. Triply-Collinear Splitting Amplitudes

In this appendix, I evaluate the light-cone current $J^{\rm LC}(1,2,3;P)$ in order to obtain the triple-collinear splitting amplitudes. This requires the evaluation of $\varepsilon_\mu^{(+)}(P;q)$ where P is not null. We can evaluate this expression by multiplying by $\varepsilon_\mu^{(-)}(P;q)\cdot q'$ and dividing by $\varepsilon_\mu^{(-)}(k_i;q)\cdot q'$, where q' is another null reference four-vector ($q\cdot q'\neq 0$, $q'\cdot k_i\neq 0$), and k_i is one of the momenta becoming collinear to P in the limit. The collinear form of the Schouten identity,

$$\sqrt{z_3}\,\langle 1\,2\rangle + \sqrt{z_1}\,\langle 2\,3\rangle - \sqrt{z_2}\,\langle 1\,3\rangle = 0\,, \tag{I.1}$$

where z_i are the momentum fractions of the k_i ($z_1+z_2+z_3=1$), and its partner with the bracket product are useful in simplifying the expressions.

Recall that the ordinary collinear splitting amplitudes [26] are,

$$
\begin{aligned}
\mathrm{C}_{+}^{\mathrm{tree}}(1^{+},2^{+};z) &= 0\,,\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{-},2^{+};z) &= \frac{z^{2}}{\sqrt{z(1-z)}\,\langle 1\,2\rangle}\,,\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{+},2^{-};z) &= -\,\mathrm{C}_{+}^{\mathrm{tree}}(2^{-},1^{+};1-z)\\
&= \frac{(1-z)^{2}}{\sqrt{z(1-z)}\,\langle 1\,2\rangle}\,,\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{-},2^{-};z) &= -\frac{1}{\sqrt{z(1-z)}\,[1\,2]}\,,
\end{aligned}
\tag{I.2}
$$

The triple-collinear splitting amplitudes are,

$$
\begin{aligned}
\mathrm{C}_{+}^{\mathrm{tree}}(1^{+},2^{+},3^{+};z_1,z_2) &= 0\,,\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{-},2^{+},3^{+};z_1,z_2) &= \frac{z_1^{2}}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle}\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{+},2^{-},3^{+};z_1,z_2) &= \frac{z_2^{2}}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle}\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{+},2^{+},3^{-};z_1,z_2) &= \mathrm{C}_{+}^{\mathrm{tree}}(3^{-},2^{+},1^{+};z_3,z_2)\\
&= \frac{z_3^{2}}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle}\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{-},2^{-},3^{+};z_1,z_2) &= -\frac{\langle 1\,2\rangle\,(\sqrt{z_1}\,[1\,3]+\sqrt{z_2}\,[2\,3])^{2}}{\langle 2\,3\rangle\,[1\,2]\,[2\,3]\,t_{123}} - \frac{z_1 z_3}{(1-z_1)\,\langle 2\,3\rangle\,[2\,3]} - \frac{(1-z_3)^{2}}{\sqrt{z_1 z_3}\,\langle 2\,3\rangle\,[1\,2]}\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{-},2^{+},3^{-};z_1,z_2) &= -\frac{\langle 1\,3\rangle^{2}\,(\sqrt{z_1}\,[1\,2]-\sqrt{z_3}\,[2\,3])^{2}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[1\,2]\,[2\,3]\,t_{123}} - \frac{\sqrt{z_2}\,\langle 1\,3\rangle\,(z_1^{3/2}\,[1\,2]+z_3^{3/2}\,[2\,3])}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[1\,2]\,[2\,3]}\\
&\qquad + \frac{z_2 z_3}{(1-z_3)\,\langle 1\,2\rangle\,[1\,2]} + \frac{z_1 z_2}{(1-z_1)\,\langle 2\,3\rangle\,[2\,3]}\\
&= -\frac{\langle 1\,3\rangle^{2}\,(\sqrt{z_1}\,[1\,2]-\sqrt{z_3}\,[2\,3])^{2}}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[1\,2]\,[2\,3]\,t_{123}} - \frac{z_3^{2}}{\sqrt{z_1 z_3}\,\langle 2\,3\rangle\,[1\,2]} - \frac{z_1^{2}}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,[2\,3]}\\
&\qquad - \frac{z_1 z_3}{(1-z_3)\,\langle 1\,2\rangle\,[1\,2]} - \frac{z_1 z_3}{(1-z_1)\,\langle 2\,3\rangle\,[2\,3]}\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{+},2^{-},3^{-};z_1,z_2) &= \mathrm{C}_{+}^{\mathrm{tree}}(3^{-},2^{-},1^{+};z_3,z_2)\\
&= -\frac{\langle 2\,3\rangle\,(\sqrt{z_2}\,[1\,2]+\sqrt{z_3}\,[1\,3])^{2}}{\langle 1\,2\rangle\,[1\,2]\,[2\,3]\,t_{123}} - \frac{z_1 z_3}{(1-z_3)\,\langle 1\,2\rangle\,[1\,2]} - \frac{(1-z_1)^{2}}{\sqrt{z_1 z_3}\,\langle 1\,2\rangle\,[2\,3]}\,,\\
\mathrm{C}_{+}^{\mathrm{tree}}(1^{-},2^{-},3^{-};z_1,z_2) &= \frac{1}{\sqrt{z_1 z_3}\,[1\,2]\,[2\,3]}
\end{aligned}
\tag{I.3}
$$

The relations between the splitting amplitudes listed above follow from the reflection properties of the Berends–Giele current. The remaining amplitudes can be obtained by parity. The photon decoupling identity leads to the following relations,

$$
\mathrm{C}_{+}^{\mathrm{tree}}(1^{\sigma_1},2^{\sigma_2},3^{\sigma_3};z_1,z_2) + \mathrm{C}_{+}^{\mathrm{tree}}(2^{\sigma_2},1^{\sigma_1},3^{\sigma_3};z_2,z_1) + \mathrm{C}_{+}^{\mathrm{tree}}(2^{\sigma_2},3^{\sigma_3},1^{\sigma_1};z_2,z_3) = 0\,, \tag{I.4}
$$

whose validity for the expressions in eqn. (I.3) is left to the reader.

In the strongly-ordered limit $s_{12} \ll s_{13}, s_{23}, t_{123}$, we expect the factorization

$$\mathrm{C}^{\rm tree}_{\sigma_P}(1^{\sigma_1},2^{\sigma_2},3^{\sigma_3};z_1,z_2) \longrightarrow \sum_{\rho=\pm} \mathrm{C}^{\rm tree}_{\sigma_P}((1{+}2)^{\rho},3^{\sigma_3};z_1+z_2)\,\mathrm{C}^{\rm tree}_{-\rho}(1^{\sigma_1},2^{\sigma_2};\tfrac{z_1}{z_1+z_2})+\cdots\,, \tag{I.5}$$

where the omitted terms are less singular.

For some of the splitting amplitudes, this limit is straightforward ($R = k_1 + k_2$, $z_R = z_1 + z_2$):

$$\begin{aligned}\mathrm{C}^{\rm tree}_{+}(1^+,2^-,3^+;z_1,z_2) &\to \frac{(1-\frac{z_1}{z_1+z_2})^2}{\sqrt{\frac{z_1}{z_1+z_2}(1-\frac{z_1}{z_1+z_2})}\,\langle 1\,2\rangle}\frac{z_R^2}{\sqrt{z_R(1-z_R)}\,\langle R\,3\rangle}\\ &= \mathrm{C}^{\rm tree}_{+}(R^-,3^+;z_R)\,\mathrm{C}^{\rm tree}_{+}(1^+,2^-;\tfrac{z_1}{z_1+z_2})\,.\end{aligned} \tag{I.6}$$

For others, it is more delicate, because we must use the Schouten identities (I.1) to reduce the strength of the leading $1/(\langle 1\,2\rangle\,[1\,2])$ pole before applying limiting relations for $k_1 \parallel k_2$,

$$\begin{aligned}
\mathrm{C}^{\rm tree}_{+}(1^-,2^+,3^-;z_1,z_2) \longrightarrow & -\frac{z_3\,\langle 1\,3\rangle^2\,[2\,3]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[1\,2]\,s_{R3}} - \frac{z_3\sqrt{z_2}\,\langle 1\,3\rangle}{\sqrt{z_1}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[1\,2]} + \frac{z_2 z_3}{(1{-}z_3)\,\langle 1\,2\rangle\,[1\,2]}\\
&+\frac{2\sqrt{z_1 z_3}\,\langle 1\,3\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,s_{R3}} - \frac{z_1\sqrt{z_2}\,\langle 1\,3\rangle}{\sqrt{z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[2\,3]}+\cdots\\
= & -\frac{z_1 z_3\,\langle 2\,3\rangle\,[2\,3]}{z_2\,\langle 1\,2\rangle\,[1\,2]\,s_{R3}} - \frac{z_3}{\langle 1\,2\rangle\,[1\,2]} + \frac{z_2 z_3}{(1{-}z_3)\,\langle 1\,2\rangle\,[1\,2]}\\
&-\frac{2\sqrt{z_1}z_3^{3/2}\,[2\,3]}{z_2\,[1\,2]\,s_{R3}} - \frac{z_3^{3/2}}{\sqrt{z_1}\,\langle 2\,3\rangle\,[1\,2]}\\
&+\frac{2\sqrt{z_1 z_3}\,\langle 1\,3\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,s_{R3}} - \frac{z_1\sqrt{z_2}\,\langle 1\,3\rangle}{\sqrt{z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[2\,3]}+\cdots\\
= & -\frac{z_1 z_3\,\langle 2\,3\rangle\,[2\,3]}{z_2\,\langle 1\,2\rangle\,[1\,2]\,s_{R3}} + \frac{z_1 z_3\,\langle 2\,3\rangle\,[2\,3]}{(1{-}z_3)\,\langle 1\,2\rangle\,[1\,2]\,s_{R3}} + \frac{z_1 z_3\,\langle 1\,3\rangle\,[1\,3]}{(1{-}z_3)\,\langle 1\,2\rangle\,[1\,2]\,s_{R3}}\\
&-\frac{2\sqrt{z_1}z_3^{3/2}\,[2\,3]}{z_2\,[1\,2]\,s_{R3}} - \frac{z_3^{3/2}}{\sqrt{z_1}\,\langle 2\,3\rangle\,[1\,2]}\\
&+\frac{2\sqrt{z_1 z_3}\,\langle 1\,3\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,s_{R3}} - \frac{z_1\sqrt{z_2}\,\langle 1\,3\rangle}{\sqrt{z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[2\,3]}+\cdots\\
= & -\frac{z_1 z_3\,\langle 2\,3\rangle\,[2\,3]}{z_2\,\langle 1\,2\rangle\,[1\,2]\,s_{R3}} + \frac{z_1 z_3\,\langle 2\,3\rangle\,[2\,3]}{(z_1+z_2)\,\langle 1\,2\rangle\,[1\,2]\,s_{R3}} + \frac{z_1^2 z_3\,\langle 2\,3\rangle\,[2\,3]}{z_2(z_1+z_2)\,\langle 1\,2\rangle\,[1\,2]\,s_{R3}}\\
&-\frac{2\sqrt{z_1}z_3^{3/2}\,[2\,3]}{z_2\,[1\,2]\,s_{R3}} - \frac{z_3^{3/2}}{\sqrt{z_1}\,\langle 2\,3\rangle\,[1\,2]} + \frac{z_1^{3/2}z_3^{3/2}\,[2\,3]}{z_2(z_1+z_2)\,[1\,2]\,s_{R3}}\\
&+\frac{2\sqrt{z_1 z_3}\,\langle 1\,3\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,s_{R3}} - \frac{z_1\sqrt{z_2}\,\langle 1\,3\rangle}{\sqrt{z_3}\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,[2\,3]} + \frac{z_1^{3/2}z_3^{3/2}\,\langle 2\,3\rangle}{z_2(z_1+z_2)\,\langle 1\,2\rangle\,s_{R3}}+\cdots
\end{aligned} \tag{I.7}$$

The coefficient of the leading pole cancels, so that we can now use the limiting relations $\langle 2\,3\rangle \to$

$\sqrt{\frac{z_2}{z_1+z_2}}\,\langle R\,3\rangle$, etc., since the corrections will give rise to terms non-singular as $s_{12} \to 0$:

$$
\begin{aligned}
&\mathrm{C}_{+}^{\rm tree}(1^-,2^+,3^-;z_1,z_2) \longrightarrow \\
&\qquad \frac{2\sqrt{z_1}z_3^{3/2}}{\sqrt{z_2(z_1+z_2)}\,[1\,2]\,\langle R\,3\rangle} - \frac{z_3^{3/2}\sqrt{z_1+z_2}}{\sqrt{z_1z_2}\,[1\,2]\,\langle R\,3\rangle} - \frac{z_1^{3/2}z_3^{3/2}}{\sqrt{z_2}(z_1+z_2)^{3/2}\,[1\,2]\,\langle R\,3\rangle} \\
&\qquad - \frac{2\sqrt{z_3}z_1^{3/2}}{\sqrt{z_2(z_1+z_2)}\,\langle 1\,2\rangle\,[R\,3]} - \frac{z_1^{3/2}\sqrt{z_1+z_2}}{\sqrt{z_2z_3}\,\langle 1\,2\rangle\,[R\,3]} - \frac{z_1^{3/2}z_3^{3/2}}{\sqrt{z_2}(z_1+z_2)^{3/2}\,\langle 1\,2\rangle\,[R\,3]} + \cdots \\
&\qquad = -\frac{z_2^2z_3^2}{\sqrt{z_1z_2z_3}(z_1+z_2)^{3/2}\,[1\,2]\,\langle R\,3\rangle} - \frac{z_1^2}{\sqrt{z_1z_2z_3}(z_1+z_2)^{3/2}\,\langle 1\,2\rangle\,[R\,3]} + \cdots \\
&\qquad = \mathrm{C}_{-}^{\rm tree}(1^-,2^+;\tfrac{z_1}{z_1+z_2})\,\mathrm{C}_{+}^{\rm tree}(R^+,3^-;z_R) + \mathrm{C}_{+}^{\rm tree}(1^-,2^+;\tfrac{z_1}{z_1+z_2})\,\mathrm{C}_{+}^{\rm tree}(R^-,3^-;z_R) + \cdots
\end{aligned}
\tag{I.8}
$$

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